RAY CATEGORIES, II

BASED ON THE TALK BY DIETER VOSSIECK

Definition.

Let X be a ray category. By the *homotopy relation* in the set of the walks in the quiver QX we mean the equivalence relation $\approx$ generated by the following conditions:

(1) $\alpha\alpha^- \approx \mathbf{1}_y$ and $\alpha^-\alpha \approx \mathbf{1}_x$ for each arrow $\alpha : x \to y$ in the quiver QX,

(2) $u \approx v$ and $u^- \approx v^-$ for each contour (u, v) of the category X,

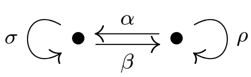
(3) if $v \approx v'$, then $wvu \approx wv'u$ for all morphisms u, v, v', w in the category X such that the above makes sense.

Notation.

We denote by $\pi_1(X, x)$ the fundamental group of a ray category X at an object x.

Example.

If X is the path category of the quiver



modulo the relations

$$\sigma\alpha = \alpha\rho,\ \rho\beta = \beta\sigma,\ \beta\alpha = \rho^2,\ \alpha\beta = \sigma^3,$$
$$\rho^3 = 0,\ \sigma^4 = 0,\ \alpha\rho^2 = 0,\ \rho^2\beta = 0,$$

then $\pi_1(X, x) = 1$ for every object x of the category X.

Definition.

Let X be a connected ray category and x an object of the category X. We define the universal cover $\tilde{X}$ of the category X as follows. First, we define a quiver Q' whose vertices are the homotopy classes of the walks in the quiver QX starting at the object x. Next, for each arrow $\alpha : y \to z$ in the quiver QX and the homotopy class $[u]$ of a walk u starting at the object x and terminating at the object y, we have an arrow $[u] \to [\alpha u]$. If R' denotes the kernel of the canonical functor $PQ' \to X$, then $\tilde{X} := PQ'/R'$.

Remark.

If X is a ray category, then the universal cover $\tilde{X}$ is the ray category.

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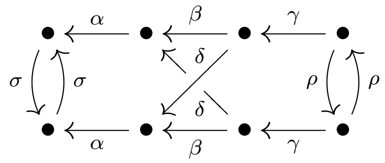
Example.

If X is the path category of the quiver



modulo the relations

$$\sigma\alpha\delta = \alpha\beta,\ \beta\gamma\rho = \delta\gamma,\ \sigma^2 = 0,\ \rho^2 = 0,$$

then the universal cover $\tilde{X}$ is not interval finite.

Definition.

By a *0-simplex* of a ray category X we mean every object of the category X.

Definition.

If $n \in \mathbb{N}_+$, then by an *n-simplex* of a ray category X we mean every sequence $(\mu_n, \ldots, \mu_1)$ of composable morphisms in the category X such that $\mu_n \cdots \mu_1 \neq 0$.

Definition.

For a ray category X we define the *simplicial complex* S_*X as follows. First, for $n \in \mathbb{N}$ we define S_nX as the set of the n-simplices of the category X. Next, for each $n \in \mathbb{N}_+$ we define face operators $d_0, \ldots, d_n : S_nX \to S_{n-1}X$ by the following formulas: if $n = 0$, then

$$d_0(\mu) := t\mu \qquad \text{and} \qquad d_1(\mu) := s\mu,$$

otherwise we put

$$d_i(\mu_n, \ldots, \mu_1) := \begin{cases} (\mu_n, \ldots, \mu_2) & i = 0, \\ (\mu_n, \ldots, \mu_{i+2}, \mu_{i+1}\mu_i, \mu_{i-1}, \ldots, \mu_1) & i \in [1, n-1], \\ (\mu_{n-1}, \ldots, \mu_1) & i = n. \end{cases}$$

Remark.

Let K be a finite simplicial complex. If X is the poset of simplicies of the complex K (ordered by the inclusion), which can be viewed as a ray category, then the simplical complex S_*X is the first baricentric subdivision of the complex K.

Remark.

If X is a ray category, then we associate with the simplicial complex S_*X the chain complex C_*X in the usual way, and use it in order to define the homologies H_*X and the cohomologies $H^*(X, G)$ of the category X for a group G.