RELATIVE HOMOLOGICAL ALGEBRA AND GORENSTEIN ALGEBRAS

BASED ON THE TALKS BY ØVIND SOLBERG

Throughout this presentation we assume that all considered algebras are Artin algebras and all considered modules are finitely generated ones.

1. Motivation

In this section we present results which serve as a motivation for our studies. More precisely, relative homological algebra is a common setup explaining connections between Λ and Γ in the situations described below.

Auslander correspondence. Let Γ be an algebra and

$$0 \to \Gamma \to I^0 \to I^1 \to I^2 \to \cdots$$

be the minimal injective resolution of Γ in $\operatorname{mod}\Gamma$. If $l \in \mathbb{N}$, then we write $\operatorname{dom.dim}\Gamma \geq l$ if I^i is injective for each $i \in [0, l-1]$ and call $\operatorname{dom.dim}\Gamma$ the dominant dimension of Γ. We say that Γ is an Auslander algebra if

$$\operatorname{dom.dim}\Gamma \geq 2 \geq \operatorname{gl.dim}\Gamma.$$

Auslander has proved that there is a bijection between the Morita equivalence classes of the representation finite algebras and the Morita equivalence classes of the Auslander algebras. This bijection is induced by the assignment

$$\Lambda \mapsto \operatorname{End}_\Lambda(M),$$

where, for a representation finite algebra Λ, M is an additive generator of $\operatorname{mod}\Lambda$. It is known that if Λ is a representation finite algebra, M is an additive generator of $\operatorname{mod}\Lambda$, and $\Gamma := \operatorname{End}_\Lambda(M)$, then $\mathcal{F}_M$ induces a duality between $\operatorname{mod}\Lambda$ and $\operatorname{proj}\Gamma$, where for a module M over an algebra Λ we put $\mathcal{F}_M := \operatorname{Hom}_\Lambda(-, M)$.

Cotilting. A module T over an algebra Λ is called cotilting provided $\operatorname{Ext}^i_\Lambda(T, T) = 0$ for each $i \in \mathbb{N}_+$, $\operatorname{id}_\Lambda T < \infty$, and there exists an exact sequence of the form

$$0 \to T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to \mathrm{D}(\Lambda) \to 0$$

such that $T_i \in \operatorname{add} T$ for each $i \in [0, n]$. If T is a cotilting module over an algebra Λ and $\Gamma := \operatorname{End}_\Lambda(T)$, then $\mathcal{F}_T(\Lambda)$ is a cotilting module

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over Γ and $\mathcal{F}_T$ induces a duality between ${}^{\perp}T$ and ${}^{\perp}\mathcal{F}_T(\Lambda)$. Here, for a subcategory $\mathcal{X}$ of the category of modules over an algebra Λ we put

$$ {}^{\perp}\mathcal{X} := \{M \in \operatorname{mod}\Lambda : \operatorname{Ext}^i_{\Lambda}(M, \mathcal{X}) = 0 \text{ for all } i \in \mathbb{N}_+\} $$

and

$$ \mathcal{X}^{\perp} := \{M \in \operatorname{mod}\Lambda : \operatorname{Ext}^i_{\Lambda}(\mathcal{X}, M) = 0 \text{ for all } i \in \mathbb{N}_+\}. $$

Moreover, if $M \in \operatorname{mod}\Lambda$ then ${}^{\perp}M := {}^{\perp}(\operatorname{add} M)$.

Auslander generator. Let Λ be an algebra. A Λ-module M is called a generator-cogenerator of $\operatorname{mod}\Lambda$, if $\Lambda \oplus \mathrm{D}(\Lambda) \in \operatorname{add} M$. By the representation dimension $\operatorname{rep.dim}\Lambda$ of Λ we mean

$$ \min\{\operatorname{gl.dim}\operatorname{End}_{\Lambda}(M) : M \text{ is a generator-cogenerator of } \operatorname{mod}\Lambda\}. $$

Iyama has proved that $\operatorname{rep.dim}\Lambda < \infty$. A generator-cogenerator M of $\operatorname{mod}\Lambda$ is said to be an Auslander generator if

$$ \operatorname{rep.dim}\Lambda = \operatorname{gl.dim}\operatorname{End}_{\Lambda}(M). $$

It is known that if M is an Auslander generator of $\operatorname{mod}\Lambda$, then $\mathcal{F}_M$ induces a duality between $\operatorname{mod}\Lambda$ and $\mathcal{F}_M(\operatorname{mod}\Lambda)$.

2. Subfunctors of $\operatorname{Ext}^1_{\Lambda}(-,-)$

Throughout this section Λ is an algebra.

For all $A, C \in \operatorname{mod}\Lambda$ fix a subset $F(C, A)$ of $\operatorname{Ext}^1_{\Lambda}(C, A)$. We say that an exact sequence

$$ \delta : 0 \to A \to B \to C \to 0 $$

is F-exact if $[\delta] \in F(C, A)$. The collection of the F-exact sequences determines a subfunctor of $\operatorname{Ext}^1_{\Lambda}(-,-)$ if and only if it is closed under pullbacks and pushouts. If, in addition, this collection is closed under Baer sums, then $F(C, A)$ is a subgroup of $\operatorname{Ext}^1_{\Lambda}(C, A)$ for all $A, C \in \operatorname{mod}\Lambda$.

A subfunctor F of $\operatorname{Ext}^1_{\Lambda}(-,-)$ is called additive if $F(C, A)$ is a subgroup of $\operatorname{Ext}^1_{\Lambda}(C, A)$ and the functors $F(C,-)$ and $F(-,A)$ are additive for all $A, C \in \operatorname{mod}\Lambda$.

Lemma 2.1. *Let F be a subfunctor of $\operatorname{Ext}^1_{\Lambda}(-,-)$. Then F is additive if and only if the collection of the F-exact sequences is closed under direct sums.*

It follows from the above lemma that if we fix a subset $F(C, A)$ of $\operatorname{Ext}^1_{\Lambda}(C, A)$ for all $A, C \in \operatorname{mod}\Lambda$, then the collection of the F-exact sequences determines an additive subfunctor of $\operatorname{Ext}^1_{\Lambda}(-,-)$ if and only if this collection is closed under pullbacks, pushouts and direct sums. Moreover, if F is an additive subfunctor of $\operatorname{Ext}^1_{\Lambda}(-,-)$, δ and δ' are exact sequences, then δ and δ' are F-exact if and only if $\delta \oplus \delta'$ is F-exact.

For the rest of the section F is an additive subfunctor of $\operatorname{Ext}^1_{\Lambda}(-,-)$.

By an F-epimorphism (F-monomorphism) we mean every epimorphism $f : B \to C$ (monomorphism $g : A \to B$, respectively) such that the sequence

$$0 \to \operatorname{Ker} f \to B \xrightarrow{f} C \to 0$$
$$(0 \to A \xrightarrow{g} B \to \operatorname{Coker} g \to 0, \text{ respectively})$$

is F-exact. A Λ-module P is called F-projective (F-injective) provided $\operatorname{Hom}_\Lambda(P, f)$ ($\operatorname{Hom}_\Lambda(g, I)$, respectively) is an epimorphism for each F-epimorphism f (F-monomorphism g, respectively). By $\mathcal{P}(F)$ ($\mathcal{I}(F)$) we denote the full subcategory of $\operatorname{mod}\Lambda$ consisting of the F-projective (F-injective, respectively) modules. We say that F has enough projectives (injectives) if for each $C \in \operatorname{mod}\Lambda$ ($A \in \operatorname{mod}\Lambda$, respectively) there exists an F-epimorphism $f : P \to C$ (F-monomorphism $g : A \to I$, respectively) such that $P \in \mathcal{P}(F)$ ($I \in \mathcal{I}(F)$, respectively).

Proposition 2.2. *If*

$$0 \to A \to B \to C \to 0$$

is an F-exact sequence and $M \in \operatorname{mod}\Lambda$, then the sequences

$$0 \to \operatorname{Hom}_\Lambda(C, M) \to \operatorname{Hom}_\Lambda(B, M) \to \operatorname{Hom}_\Lambda(A, M) \to F(C, M)$$

and

$$0 \to \operatorname{Hom}_\Lambda(M, A) \to \operatorname{Hom}_\Lambda(M, B) \to \operatorname{Hom}_\Lambda(M, A) \to F(M, A)$$

are exact for each $M \in \operatorname{mod}\Lambda$.

Corollary 2.3.

(a) *Let $P \in \operatorname{mod}\Lambda$. Then $P \in \mathcal{P}(F)$ if and only if $F(P, A) = 0$ for each $A \in \operatorname{mod}\Lambda$.*

(b) *Let $I \in \operatorname{mod}\Lambda$. Then $I \in \mathcal{I}(F)$ if and only if $F(C, I) = 0$ for each $C \in \operatorname{mod}\Lambda$.*

Given an exact sequence

$$\delta : 0 \to A \to B \to C \to 0$$

one defines the defect functors δ^* and δ_* such that the sequences

$$0 \to \operatorname{Hom}_\Lambda(-, A) \to \operatorname{Hom}_\Lambda(-, B) \to \operatorname{Hom}_\Lambda(-, C) \to \delta^* \to 0$$

and

$$0 \to \operatorname{Hom}_\Lambda(C, -) \to \operatorname{Hom}_\Lambda(B, -) \to \operatorname{Hom}_\Lambda(A, -) \to \delta_* \to 0$$

are exact. Auslander has showed that the functors $\mathrm{D} \circ \delta^*$ and $\delta_* \circ \mathrm{D}\,\mathrm{Tr}$ are isomorphic. Using this fact one shows the following.

Lemma 2.4. *We have*

$$\mathcal{I}(F) = \operatorname{inj}\Lambda \vee \mathrm{D}\,\mathrm{Tr}(\mathcal{P}(F)) \qquad \textit{and} \qquad \mathcal{P}(F) = \operatorname{proj}\Lambda \vee \mathrm{Tr}\,\mathrm{D}(\mathcal{I}(F)).$$

Lemma 2.5.

(a) *Assume that F has enough projectives. If δ is an exact sequence, then δ is F-exact if and only if $\delta^*(P) = 0$ for each $P \in \mathcal{P}(F)$.*

(b) *Assume that F has enough injectives. If δ is an exact sequence, then δ is F-exact if and only if $\delta_*(I) = 0$ for each $I \in \mathcal{I}(F)$.*

For an additive subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$ and $A, C \in \operatorname{mod}\Lambda$ we put

$$F_{\mathcal{X}}(C, A) := \{\eta \in \operatorname{Ext}^1_\Lambda(C, A) : \operatorname{Hom}_\Lambda(X, \eta) \text{ is exact for each } X \in \mathcal{X}\}$$

and

$$F^{\mathcal{X}}(C, A) := \{\eta \in \operatorname{Ext}^1_\Lambda(C, A) : \operatorname{Hom}_\Lambda(\eta, X) \text{ is exact for each } X \in \mathcal{X}\}.$$

Observe that $F_{\operatorname{proj} A} = \operatorname{Ext}^1_\Lambda(-,-) = F^{\operatorname{inj}\Lambda}$, while $F_{\operatorname{mod}\Lambda} = 0 = F^{\operatorname{mod}\Lambda}$.

Proposition 2.6. *Let $\mathcal{X}$ be an additive subcategory of $\operatorname{mod}\Lambda$. Then $F_{\mathcal{X}}$ and $F^{\mathcal{X}}$ are additive subfunctors of $\operatorname{Ext}^1_\Lambda(-,-)$, and $F_{\mathcal{X}} = F^{\operatorname{D}\operatorname{Tr}(\mathcal{X})}$ and $F^{\mathcal{X}} = F_{\operatorname{Tr}\operatorname{D}(\mathcal{X})}$.*

Approximations (precovers/preenvelopes). Let $\mathcal{X}$ be an additive subcategory of $\operatorname{mod}\Lambda$. By a right $\mathcal{X}$-approximation of $C \in \operatorname{mod}\Lambda$ (left $\mathcal{X}$-approximation of $A \in \operatorname{mod}\Lambda$) we mean every $f \in \operatorname{Hom}_\Lambda(X, C)$ ($g \in \operatorname{Hom}_\Lambda(A, X)$, respectively) such that $X \in \mathcal{X}$ and $\operatorname{Hom}_\Lambda(X', f)$ ($\operatorname{Hom}_\Lambda(g, X')$, respectively) is an epimorphism for each $X' \in \operatorname{mod}\Lambda$. A right $\mathcal{X}$-approximation $f \in \operatorname{Hom}_\Lambda(X, C)$ of $C \in \operatorname{mod}\Lambda$ (left $\mathcal{X}$-approximation $g \in \operatorname{Hom}_\Lambda(A, X)$ of $A \in \operatorname{mod}\Lambda$) is called minimal if every $h \in \operatorname{End}_\Lambda(X)$ such that $f \circ h = f$ ($h \circ g = g$, respectively) is an isomorphism. We say that $\mathcal{X}$ is contravariantly (covariantly) finite if every $C \in \operatorname{mod}\Lambda$ ($A \in \operatorname{mod}\Lambda$) has a right (left, respectively) $\mathcal{X}$-approximation. Finally, if $\mathcal{X}$ is both contravariantly and covariantly finite, then we say that $\mathcal{X}$ is functorially finite. Examples of functorially finite subcategories of $\operatorname{mod}\Lambda$ are $\operatorname{add} M$, $\operatorname{Fac} M$ and $\operatorname{Sub} M$ for $M \in \operatorname{mod}\Lambda$.

Theorem 2.7.

(a) *F has enough projectives if and only if $\mathcal{P}(F)$ is contravariantly finite and $F = F_{\mathcal{P}(F)}$.*

(b) *F has enough injectives if and only if $\mathcal{I}(F)$ is covariantly finite and $F = F^{\mathcal{I}(F)}$.*

For $M \in \operatorname{mod}\Lambda$ we put

$$F_M := F_{\operatorname{add} M} \qquad \text{and} \qquad F^M := F^{\operatorname{add} M}.$$

Let $M \in \operatorname{mod}\Lambda$. Then

$$\mathcal{P}(F_M) = \operatorname{add}(\Lambda \oplus M) \qquad \text{and} \qquad \mathcal{I}(F^M) = \operatorname{add}(\operatorname{D}(\Lambda) \oplus M).$$

Obviously,

$$F_M = F_{\Lambda \oplus M} \qquad \text{and} \qquad F^M := F^{\operatorname{D}(\Lambda)\oplus M}.$$

Moreover,

$$F_M = F^{\mathrm{D}\,\mathrm{Tr}(M)} \qquad \text{and} \qquad F^M = F_{\mathrm{Tr}\,\mathrm{D}(M)}.$$

In particular, both F_M and F^M have enough projectives and injectives.

3. Relative homology

Throughout this section we assume that Λ is an algebra and F is an additive subfunctor of $\mathrm{Ext}^1_\Lambda(-,-)$ with enough projectives and injectives.

A long exact sequence

$$\cdots \to M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \to \cdots$$

is said to be F-exact if the sequence

$$0 \to \mathrm{Ker}\, d_n \to M_n \to \mathrm{Im}\, d_n \to 0$$

is F-exact for each $n \in \mathbb{Z}$. Since F has enough projectives, for each $C \in \mathrm{mod}\,\Lambda$ there exists an F-projective resolution, i.e. an F-exact sequence of the form

$$\cdots \to P_2 \to P_1 \to P_0 \to C \to 0$$

such that $P_i \in \mathcal{P}(F)$ for each $i \in \mathbb{N}$. Similarly, since F has enough injectives, for each $A \in \mathrm{mod}\,\Lambda$ there exists an F-injective resolution, i.e. an F-exact sequence of the form

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

such that $I^i \in \mathcal{I}(F)$ for each $i \in \mathbb{N}$. If $i \in \mathbb{N}$, $A, C \in \mathrm{mod}\,\Lambda$, and

$$\cdots \to P_2 \to P_1 \to P_0 \to C \to 0$$

is an F-projective resolution of C, then we denote by $\mathrm{Ext}^i_F(C, A)$ the i-th homology of the complex

$$0 \to \mathrm{Hom}_\Lambda(P_0, A) \to \mathrm{Hom}_\Lambda(P_1, A) \to \mathrm{Hom}_\Lambda(P_2, A) \to \cdots .$$

One shows that this definition does not depend on the choice of a projective resolution of C and $\mathrm{Ext}^i_F(C, A)$ is isomorphic to the i-th homology of the complex

$$0 \to \mathrm{Hom}_\Lambda(C, I^0) \to \mathrm{Hom}_\Lambda(C, I^1) \to \mathrm{Hom}_\Lambda(C, I^2) \to \cdots ,$$

where

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

is an F-injective resolution of A. One easily checks that

$$\mathrm{Ext}^0_F(C, A) = \mathrm{Hom}_\Lambda(C, A) \qquad \text{and} \qquad \mathrm{Ext}^1_F(C, A) = F(C, A).$$

Moreover, if

$$0 \to A \to B \to C \to 0$$

is an F-exact sequence, then for each $X \in \operatorname{mod}\Lambda$ we have long exact sequences

$$\begin{aligned}0 \to \operatorname{Hom}_\Lambda(X,A) \to \operatorname{Hom}_\Lambda(X,B) \to \operatorname{Hom}_\Lambda(X,C) \to \\ F(X,A) \to F(X,B) \to F(X,C) \to \\ \operatorname{Ext}^2_F(X,A) \to \operatorname{Ext}^2_F(X,B) \to \operatorname{Ext}^2_F(X,C) \to \cdots\end{aligned}$$

and

$$\begin{aligned}0 \to \operatorname{Hom}_\Lambda(C,X) \to \operatorname{Hom}_\Lambda(B,X) \to \operatorname{Hom}_\Lambda(A,X) \to \\ F(C,X) \to F(B,X) \to F(A,X) \to \\ \operatorname{Ext}^2_F(C,X) \to \operatorname{Ext}^2_F(B,X) \to \operatorname{Ext}^2_F(A,X) \to \cdots .\end{aligned}$$

By the F-projective dimension $\operatorname{pd}_F C$ of $C \in \operatorname{mod}\Lambda$ we mean the minimal $n \in \mathbb{N}$ such that there exists an F-projective resolution of C of the form

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to C \to 0,$$

where by definition $\min\varnothing := \infty$. One shows that if $n \in \mathbb{N}$, then $\operatorname{pd}_F C \leq n$ if and only if $\operatorname{Ext}^{n+1}_F(C,-) = 0$. Dually, by the F-injective dimension $\operatorname{id}_F A$ of $A \in \operatorname{mod}\Lambda$ we mean the minimal $n \in \mathbb{N}$ such that there exists an F-injective resolution of A of the form

$$0 \to A \to I^0 \to I^1 \to \cdots \to I^{n-1} \to I^n \to 0.$$

Again, if $n \in \mathbb{N}$, then $\operatorname{id}_F A \leq n$ if and only if $\operatorname{Ext}^{n+1}_F(-,A) = 0$. Finally, we put

$$\operatorname{gl.dim}_F \Lambda := \sup\{\operatorname{pd}_F C : C \in \operatorname{mod}\Lambda\}.$$

Obviously,

$$\operatorname{gl.dim}_F \Lambda = \sup\{\operatorname{id}_F A : A \in \operatorname{mod}\Lambda\}.$$

For $A, C \in \operatorname{mod}\Lambda$ let $\mathcal{P}(F)(A,C)$ be the subspace of $\operatorname{Hom}_\Lambda(A,C)$ consisting of the homomorphisms which factor through a module from $\mathcal{P}(F)$, and

$$\underline{\operatorname{Hom}}_{\mathcal{P}(F)}(C,A) := \operatorname{Hom}_\Lambda(C,A)/\mathcal{P}(F)(C,A).$$

Proposition 3.1. *If $A, C \in \operatorname{mod}\Lambda$, then we have a functorial isomorphism*

$$\operatorname{Ext}^1_F(C, \operatorname{D}\operatorname{Tr}(A)) \simeq \operatorname{D}(\underline{\operatorname{Hom}}_{\mathcal{P}(F)}(A,C)).$$

4. Relative cotilting modules

Throughout this section we assume that Λ is an algebra and F is an additive subfunctor of $\operatorname{Ext}^1_\Lambda(-,-)$ with enough projectives and injectives.

A Λ-module T is called F-cotilting if $\mathrm{Ext}^i_F(T,T)=0$ for each $i\in\mathbb{N}_+$, $\mathrm{id}_F T<\infty$, and for each $I\in\mathcal{I}(F)$ there exists an F-exact sequence of the form

$$0\to T_n\to T_{n-1}\to\cdots\to T_1\to T_0\to I\to 0$$

such that $T_i\in\mathrm{add}\,T$ for each $i\in[0,n]$. Obviously, if $F=\mathrm{Ext}^1_\Lambda(-,-)$ and $T\in\mathrm{mod}\,\Lambda$, then T is F-cotilting if and only if T is cotilting. For an F-cotilting Λ-module T we put

$$^{\perp}_{F}T:=\{X\in\mathrm{mod}\,\Lambda:\mathrm{Ext}^i_\Lambda(X,T)=0\text{ for each }i\in\mathbb{N}_+\},$$

and denote by ${}_F\mathcal{X}_T$ the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of $X\in{}^{\perp}_{F}T$ such that there exists an F-exact sequence of the form

$$0\to X\xrightarrow{f^0}T^0\xrightarrow{f^1}T^1\xrightarrow{f^2}T_2\to\cdots$$

such that $T^i\in\mathrm{add}\,T$ and $\mathrm{Im}\,f^i\in{}^{\perp}_{F}T$ for each $i\in\mathbb{N}$.

Proposition 4.1. *Let T be an F-cotilting Λ-module. Then ${}_F\mathcal{X}_T$ is closed under F-extensions and kernels of F-epimorphisms.*

An additive subcategory $\mathcal{X}$ of $\mathrm{mod}\,\Lambda$ is called F-resolving if $\mathcal{P}(F)\subseteq\mathcal{X}$, and $\mathcal{X}$ is closed under F-extensions and kernels of F-epimorphisms. For an F-resolving subcategory $\mathcal{X}$ of $\mathrm{mod}\,\Lambda$ we define the relative $\mathcal{X}$-resolution dimension $\mathcal{X}\text{-res.}\dim_F(\mathrm{mod}\,\Lambda)$ of $\mathrm{mod}\,\Lambda$ to be the minimal $n\in\mathbb{N}$ such that for each $C\in\mathrm{mod}\,\Lambda$ there exists an F-exact sequence of the form

$$0\to X_n\to X_{n-1}\to\cdots\to X_1\to X_0\to C\to 0$$

such that $X_i\in\mathcal{X}$ for each $i\in[0,n]$. If $F=\mathrm{Ext}^1_\Lambda(-,-)$, then an additive subcategory $\mathcal{X}$ of $\mathrm{mod}\,\Lambda$ is F-resolving if and only if $\mathcal{X}$ is resolving, and $\mathcal{X}\text{-res.}\dim_F(\mathrm{mod}\,\Lambda)$ is the $\mathcal{X}$-resolution dimension $\mathcal{X}\text{-res.}\dim(\mathrm{mod}\,\Lambda)$ of $\mathrm{mod}\,\Lambda$.

Theorem 4.2. *If T is an F-cotilting module, then ${}_F\mathcal{X}_T={}^{\perp}_{F}T$, ${}_F\mathcal{X}_T$ is an F-resolving contravariantly finite subcategory of* $\mathrm{mod}\,\Lambda$*, and*

$${}_F\mathcal{X}_T\text{-res.}\dim_F(\mathrm{mod}\,\Lambda)<\infty.$$

Proof. We first show that ${}_F\mathcal{X}_T={}^{\perp}_{F}T$. Obviously ${}_F\mathcal{X}_T\subseteq{}^{\perp}_{F}T$, thus it remains to prove that ${}^{\perp}_{F}T\subseteq{}_F\mathcal{X}_T$. In order to do this we show that for each $C\in{}^{\perp}_{F}T$ there exists an F-exact sequence of the form

$$0\to C\to T_0\to K\to 0$$

such that $T_0\in\mathrm{add}\,T$ and $K\in{}^{\perp}_{F}T$.

Let $C\in{}^{\perp}_{F}T$. Since F has enough injectives, there exists an F-exact sequence of the form

$$0\to C\to I\to C_0\to 0$$

such that $I\in\mathcal{I}(F)$. Next, there exists an F-exact sequence of the form

$$0\to T_n\xrightarrow{d_n}T_{n-1}\to\cdots\to T_1\xrightarrow{d_1}T_0\xrightarrow{d_0}I\to 0$$

such that $T_i \in \operatorname{add} T$ for each $i \in [0, n]$. If $L := \operatorname{Ker} d_0$, then

$$\operatorname{Ext}^1_F(C, L) \simeq \operatorname{Ext}^n_F(C, T^n) = 0.$$

Consequently, if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & T_0 & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

is the pullback diagram, then the upper sequence splits. Thus there exists an exact sequence δ of the form

$$0 \to C \xrightarrow{g} T_0 \to K \to 0,$$

where without loss of generality we may assume that g is a left $\operatorname{add} T$-approximation. Observe that we have the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & T_0 & \longrightarrow & K & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C & \longrightarrow & I & \longrightarrow & C_0 & \longrightarrow & 0 \end{array},$$

hence δ is F-exact. Next,

$$\operatorname{Ext}^1_F(K, T) \simeq \operatorname{Ext}^1_F(T_0, T) = 0,$$

since $\operatorname{Hom}_\Lambda(g, T)$ is an epimorphism. Finally,

$$\operatorname{Ext}^{n+1}_F(K, T) \simeq \operatorname{Ext}^n_F(C, T) = 0$$

for each $n \in \mathbb{N}_+$, thus $K \in {}^{\perp}_{F}T$, and this finishes the proof of the equality ${}_F\mathcal{X}_T = {}^{\perp}_{F}T$.

Now observe that $\mathcal{P}(F) \subseteq {}^{\perp}_{F}T = {}_F\mathcal{X}_T$, hence ${}_F\mathcal{X}_T$ is an F-resolving subcategory of $\operatorname{mod} \Lambda$ due to the previous proposition.

Next we show that

$${}_F\mathcal{X}_T\text{-res.}\dim_T(\operatorname{mod} \Lambda) \leq n,$$

where $n := \operatorname{id}_F T$. Indeed, if $C \in \operatorname{mod} \Lambda$, then there exists an F-exact sequence of the form

$$\cdots \to X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} C \to 0$$

such that $X_i \in \mathcal{P}(F)$ for each $i \in \mathbb{N}$. Then

$$\operatorname{Ext}^i_F(\operatorname{Ker} d_{n-1}, T) \simeq \operatorname{Ext}^{i+n}_F(C, T) = 0$$

for each $i \in \mathbb{N}_+$, hence $\operatorname{Ker} d_{n-1} \in {}^{\perp}_{F}T = {}_F\mathcal{X}_T$.

Observe that ${}_F\mathcal{X}_T$ is F-resolving, ${}_F\mathcal{X}_T\text{-res.}\dim_F(\operatorname{mod} \Lambda) < \infty$, and $\operatorname{add} T$ is an Ext_F-injective cogenerator of ${}_F\mathcal{X}_T$. This implies (analogously to a classical result of Auslander and Buchsbaum) that ${}_F\mathcal{X}_T$ is contravariantly finite. □

Lemma 4.3. *If T is an F-cotilting Λ-module and $\Gamma := \operatorname{End}_\Lambda(T)$, then $\mathcal{F}_T|_{{}_F\mathcal{X}_T}$ is fully faithful and maps F-exact sequences to exact sequences.*

Proposition 4.4. *If T is an F-cotilting Λ-module and $\Gamma := \operatorname{End}_\Lambda(T)$, then $\Lambda \simeq \operatorname{End}_\Gamma(\mathcal{F}_T(\Lambda))$ and $\mathcal{F}$ induces a functorial isomorphism*

$$\operatorname{Ext}^i_F(C, A) \simeq \operatorname{Ext}^i_\Gamma(\mathcal{F}_T(A), \mathcal{F}_T(C))$$

for all $i \in \mathbb{N}$ and $A, C \in {}_F\mathcal{X}_T$.

Proposition 4.5. *Let T be an F-cotilting Λ-module and $\Gamma := \operatorname{End}_\Lambda(T)$. Then the following hold.*

(a) *$\mathcal{F}_T({}_F\mathcal{X}_T) = {}^\perp\mathcal{F}_T(\mathcal{P}(F))$ and $\mathcal{F}_T({}_F\mathcal{X}_T)$ is a resolving subcategory of* $\operatorname{mod}\Gamma$.
(b) $\mathcal{F}_T({}_F\mathcal{X}_T)$-res. $\dim(\operatorname{mod}\Gamma) \leq \operatorname{id}_F T + 2$.
(c) *$\mathcal{F}_T(\mathcal{P}(F))$ is an* Ext*-injective cogenerator of $\mathcal{F}_T({}_F\mathcal{X}_T)$.*
(d) *$\mathcal{F}_T({}_F\mathcal{X}_T)$ is a contravariantly finite subcategory of* $\operatorname{mod}\Lambda$.

Recall that, given an algebra Γ, there is a bijection between the isomorphism classes of the basic cotilting Γ-modules and the resolving contravariantly finite subcategories $\mathcal{X}$ of $\operatorname{mod}\Gamma$ satisfying the condition that $\mathcal{X}$-res. $\dim(\operatorname{mod}\Lambda) < \infty$, induced by the assignment $T \mapsto {}^\perp T$. The inverse bijection is induced by the assignment $\mathcal{X} \mapsto T$, where T is a chosen basic additive generator of $\mathcal{X} \cap \mathcal{X}^\perp$.

Theorem 4.6. *Let T be an F-cotilting Λ-module and $\Gamma := \operatorname{End}_\Lambda(T)$. Then the following hold.*

(a) $\mathcal{F}_T(\mathcal{P}(F)) = \mathcal{F}_T({}_F\mathcal{X}_T) \cap \mathcal{F}_T({}_F\mathcal{X}_T)^\perp$.
(b) *There exists $T_0 \in \operatorname{mod}\Gamma$ such that $\operatorname{add} T_0 = \mathcal{F}_T(\mathcal{P}(F))$.*
(c) *If $\operatorname{add} T_0 = \mathcal{F}_T(\mathcal{P}(F))$ for $T_0 \in \operatorname{mod}\Gamma$, then T_0 is a cotilting Γ-module and*

$$\operatorname{id}_F T \leq \operatorname{id}_\Gamma T_0 \leq \operatorname{id}_F T + 2.$$

(d) *The number of the isomorphism classes of the indecomposable modules in $\operatorname{add} T$ equals the number of the isomorphism classes of the indecomposable modules in $\mathcal{P}(F)$. In particular, $\mathcal{P}(F)$ is of finite type.*
(e) *We have*

$$\operatorname{gl.dim}_F \Lambda - \operatorname{id}_F T \leq \operatorname{gl.dim}\Gamma \leq \operatorname{gl.dim}_F \Lambda + \operatorname{id}_F T + 2.$$

Recall that if T is a cotilting Λ-module and $\Gamma := \operatorname{End}_\Lambda(T)$, then $\operatorname{id}_\Gamma \mathcal{F}_T(\Lambda) = \operatorname{id}_\Lambda T$ and

$$\operatorname{gl.dim}\Lambda - \operatorname{id}_\Lambda T \leq \operatorname{gl.dim}\Gamma \leq \operatorname{gl.dim}\Lambda + \operatorname{id}_\Lambda T.$$

Now we return to the examples from Section 1.

First assume that Λ is representation finite and $F = 0$, and let M be an additive generator of $\operatorname{mod}\Lambda$. Then M is an F-cotilting Λ-module such that ${}_F\mathcal{X}_M = \operatorname{mod}\Lambda$. If $\Gamma := \operatorname{End}_\Lambda(T)$, then our results say that $\mathcal{F}_M$ induces a duality between $\operatorname{mod}\Lambda$ and $\operatorname{proj}\Gamma$, and $\operatorname{gl.dim}\Gamma \leq 2$.

Next assume that M is an Auslander generator of $\operatorname{mod}\Lambda$ and $F = F^M$. Then one easily checks that M is an F-cotilting Λ-module such that $\operatorname{id}_F T = 0$ and ${}_F\mathcal{X}_M = \operatorname{mod}\Lambda$. Note that

$$\mathcal{P}(F) = \operatorname{add}(\Lambda \oplus \operatorname{Tr}\operatorname{D}(M)).$$

Consequently, if

$$\Gamma := \operatorname{End}_\Lambda(T) \qquad \text{and} \qquad T_0 := \mathcal{F}_M(\Lambda \oplus \operatorname{Tr}\operatorname{D}(M)),$$

then $\mathcal{F}_M$ induces a duality between $\operatorname{mod}\Lambda$ and ${}^\perp T_0$, T_0 is a cotilting Γ-module with $\operatorname{id}_\Gamma T_0 \leq 2$, and

$$\operatorname{gl.dim}_F \Lambda \leq \operatorname{rep.dim}\Lambda = \operatorname{gl.dim}\Gamma \leq \operatorname{gl.dim}_F \Lambda + 2.$$

We conclude this section with the following analogue of Bongartz completion theorem.

Proposition 4.7. *Assume that $\mathcal{P}(F)$ is of finite type and let T be a Λ-module such that $\operatorname{Ext}^1_F(T,T) = 0$ and $\operatorname{id}_F T \leq 1$. Then there exists $X \in \operatorname{mod}\Lambda$ such that $T \oplus X$ is F-cotilting, $\operatorname{id}_F(T \oplus X) \leq 1$, and for each $I \in \mathcal{I}(F)$ there exists an exact sequence of the form*

$$0 \to T_1 \to T_0 \to I \to 0$$

with $T_0 \in \operatorname{add}(T \oplus X)$ and $T_1 \in \operatorname{add} T$. In particular, T is cotilting if and only if the number of the isomorphism classes of the indecomposable modules in $\operatorname{add} T$ equals the number of isomorphism classes of the indecomposable modules in $\mathcal{P}(F)$.

5. Derived equivalence

Throughout this section we assume that Λ is an algebra and F is an additive subfunctor of $\operatorname{Ext}^1_\Lambda(-,-)$.

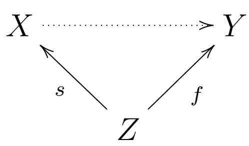
Let $\mathcal{C}$ be a triangulated category. A full subcategory $\mathcal{N}$ of $\mathcal{C}$ is called a null system if the following conditions are satisfied:

(N1) $0 \in \mathcal{N}$,
(N2) if $X \in \mathcal{N}$, then $X[1] \in \mathcal{N}$,
(N3) if

$$X \to Y \to Z \to X[1]$$

is a distinguished triangle in $\mathcal{C}$ with $X, Y \in \mathcal{N}$, then $Z \in \mathcal{N}$.

Given a null system $\mathcal{N}$ in $\mathcal{C}$ we denote by $\mathcal{S}(\mathcal{N})$ the class of all morphisms s in $\mathcal{C}$ such that $\operatorname{cone} s \in \mathcal{N}$. Moreover, by $\mathcal{C}/\mathcal{N}$ we denote the category with the same objects as $\mathcal{C}$ and with the morphisms from X to Y being the fractions of the form



,

where $s \in \mathcal{S}(\mathcal{N})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(Z, Y)$. For example, if $\mathcal{K}(\mathrm{mod}\,\Lambda)$ denotes the homotopy category of the complexes of Λ-modules and $\mathcal{N}$ is the full subcategory of $\mathcal{K}(\mathrm{mod}\,\Lambda)$ consisting of the acyclic complexes, then $\mathcal{N}$ is a null system in $\mathcal{K}(\mathrm{mod}\,\Lambda)$ and $\mathcal{K}(\mathrm{mod}\,\Lambda)/\mathcal{N}$ is the derived category $\mathcal{D}(\mathrm{mod}\,\Lambda)$ of $\mathrm{mod}\,\Lambda$.

Let $\mathcal{N}_F$ denote the full subcategory of $\mathcal{K}(\mathrm{mod}\,\Lambda)$ consisting of the F-exact complexes. Observe that $\mathcal{N}_F$ consists of the acyclic complexes if $F = \mathrm{Ext}^1_\Lambda(-,-)$. We say that F is closed if the sequence

$$F(X, A) \to F(X, B) \to F(X, C)$$

is exact for all $X \in \mathrm{mod}\,\Lambda$ and F-exact sequences

$$0 \to A \to B \to C \to 0.$$

Equivalently, F is closed if and only if the sequence

$$F(C, X) \to F(B, X) \to F(A, X)$$

is exact for all $X \in \mathrm{mod}\,\Lambda$ and each F-exact sequences

$$0 \to A \to B \to C \to 0.$$

Obviously, if F has enough projectives and injectives, then F is closed. We have the following.

Theorem 5.1 (Buan)**.** *The category $\mathcal{N}_F$ is a null system in $\mathcal{K}(\mathrm{mod}\,\Lambda)$ if and only if F is closed.*

If F is closed, then we put

$$\mathcal{D}_F(\mathrm{mod}\,\Lambda) := \mathcal{K}(\mathrm{mod}\,\Lambda)/\mathcal{N}_F$$

and we denote by $\mathcal{D}^b_F(\mathrm{mod}\,\Lambda)$ the additive subcategory of $\mathcal{D}(\mathrm{mod}\,\Lambda)$ generated by the bounded complexes.

Theorem 5.2 (Buan)**.** *Assume that F is closed. If T is an F-cotilting Λ-module and $\Gamma := \mathrm{End}_\Lambda(T)$, then $\mathcal{F}_T$ induces a duality*

$$\mathcal{D}^b_F(\mathrm{mod}\,\Lambda) \to \mathcal{D}^b(\mathrm{mod}\,\Gamma).$$

Obviously, if $F = \mathrm{Ext}^1_\Lambda(-,-)$, then we obtain a classical theorem of Happel.

6. n-Auslander algebras

Throughout this section n is a positive integer.

An algebra Γ is called an n-Auslander algebra if

$$\mathrm{dom.\,dim}\,\Gamma \geq n + 1 \geq \mathrm{gl.\,dim}\,\Gamma.$$

An additive subcategory $\mathcal{C}$ of the category of modules over an algebra Λ is called n-cluster tilting if $\mathcal{C}$ is extension closed and functorially finite, and ${}^{\perp_n}\mathcal{C} = \mathcal{C} = \mathcal{C}^{\perp_n}$, where for a subcategory $\mathcal{D}$ of the category of modules over an algebra Λ we put

$${}^{\perp_n}\mathcal{D} := \{X \in \mathrm{mod}\,\Lambda : \mathrm{Ext}^i_\Lambda(X, \mathcal{D}) = 0 \text{ for each } i \in [1, n-1]\}$$

and

$$\mathcal{D}^{\perp_n} := \{X \in \operatorname{mod}\Lambda : \operatorname{Ext}^i_\Lambda(\mathcal{D}, X) = 0 \text{ for each } i \in [1, n-1]\}.$$

By an n-cluster tilting module over an algebra Λ we mean every $M \in \operatorname{mod}\Lambda$ such that $\operatorname{add} M$ is n-cluster tilting.

Theorem 6.1 (Iyama). *The assignment*

$$(\Lambda, M) \mapsto \operatorname{End}_\Lambda(M)$$

induces a bijection between the Morita equivalence classes of the pairs (Λ, M) consisting of an algebra Λ and an n-cluster tilting Λ-module M, and the Morita equivalence classes of the n-Auslander algebras.

For an algebra Λ we put

$$\tau_n := \mathrm{D}\,\mathrm{Tr}\,\Omega^{n-1}_\Lambda \qquad \text{and} \qquad \tau_n^- := \mathrm{Tr}\,\mathrm{D}\,\Omega^{-(n-1)}_\Lambda.$$

Iyama has proved that if $\mathcal{C}$ is an n-cluster tilting subcategory of the category of modules over an algebra Λ, then τ_n and τ_n^- induce mutually quasi-inverse equivalences between $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$, and give rise to n-fold almost split extensions in $\mathcal{C}$.

At the end of this section we illustrate how the theory developed so far can be used in the proof of one of the facts constituting the above theorem.

Proposition 6.2. *If M is an n-cluster tilting module over an algebra Λ and $\Gamma := \operatorname{End}_\Lambda(M)$, then Γ is an n-Auslander algebra.*

Proof. Let $F := F^M$. Then $\operatorname{add} M = \mathcal{I}(F)$, hence M is F-cotilting. For $A \in \operatorname{mod}\Lambda$ we define Λ-modules A_i, $i \in \mathbb{N}$, in the following way: $A_0 := A$ and A_i is the cokernel of the minimal left $\operatorname{add} M$-approximation of A_{i-1} for $i \in \mathbb{N}_+$. Using Wakamatsu lemma we obtain that $\operatorname{Ext}^j_\Lambda(A_i, M) = 0$ for all $i \in \mathbb{N}$ and $j \in [1, i]$. In particular, $A_{n-1} \in {}^{\perp_n}(\operatorname{add} M) = \operatorname{add} M$, hence $\operatorname{gl.dim}_F \Lambda \leq n-1$. Consequently, $\operatorname{gl.dim}\Gamma \leq n+1$ according to Theorem 4.6(e). Moreover, $\operatorname{dom.dim}\Gamma \geq n+1$, as follows from the following theorem. □

Theorem 6.3 (Müller). *Let M be a generator-cogenerator of the category of modules over an algebra Λ such that $\operatorname{Ext}^i_\Lambda(M, M) = 0$ for all $i \in [1, n-1]$. If $\Gamma := \operatorname{End}_\Lambda(M)$, then $\operatorname{dom.dim}\Gamma \geq n+1$.*

7. D Tr-selfinjective algebras

For an algebra Λ we denote by $\mathcal{O}_\Lambda$ the additive subcategory of $\operatorname{mod}\Lambda$ generated by the modules $(\mathrm{Tr}\,\mathrm{D})^i(\Lambda)$, $i \in \mathbb{N}$. We say that an algebra Λ is D Tr-selfinjective if $\mathcal{O}_\Lambda$ is of finite type. Examples of D Tr-selfinjective algebras are the algebras of finite type, the selfinjective algebras, and the Auslander algebras of the selfinjective algebras.

For an algebra Γ we denote by $I(\Gamma)$ the maximal injective direct summand of Γ. Moreover, if $\Lambda := \operatorname{End}_\Gamma(I(\Gamma))$, then by $M(\Gamma)$ we denote the basic additive generator of the additive subcategory of $\operatorname{mod}\Lambda$ generated by the indecomposable direct summands of $\mathcal{F}_{I(\Gamma)}(\Gamma)$ which do not belong to $\mathcal{O}_\Lambda$. Recall that an algebra Γ is called Gorenstein if and only if $\operatorname{id}_\Gamma \Gamma < \infty$ and $\operatorname{pd}_\Gamma \mathrm{D}(\Gamma) < \infty$.

Theorem 7.1 (Auslander/Solberg)**.** *The assignment*

$$\Gamma \mapsto (\operatorname{End}_\Gamma(I(\Gamma)), M(\Gamma))$$

induces a bijection between the Morita equivalence classes of the Gorenstein algebras such that

$$\operatorname{dom.dim}\Gamma = 2 = \operatorname{id}_\Gamma \Gamma,$$

and the Morita equivalence classes of the pairs (Λ, M) consisting of a $\mathrm{D}\operatorname{Tr}$*-selfinjective algebra Λ and a Λ-module M such that $M \simeq \mathrm{D}\operatorname{Tr}(M)$ and either $M \neq 0$ or $\mathcal{O}_\Lambda \neq \operatorname{proj}\Lambda$.*

8. τ_n-SELFINJECTIVE ALGEBRAS

Throughout this section n is a positive integer. The aim of this section is to study the Gorenstein algebras Γ such that

$$\operatorname{dom.dim}\Gamma \geq n+1 \geq \operatorname{id}_\Gamma \Gamma.$$

A direct summand X' of a module X over an algebra Γ is called dualizing if there exists an exact sequence of the form

$$0 \to X \xrightarrow{f} X'_0 \to X'_1$$

such that f is a left $\operatorname{add} X'$-approximation of X and $X'_1 \in \operatorname{add} X'$.

Theorem 8.1 (Auslander/Solberg)**.** *Let M be a dualizing direct summand of a cotilting module T over an algebra Γ. If $\Lambda := \operatorname{End}_\Gamma(M)$ and $F := F_{\mathcal{F}_M(T)}$, then $\mathcal{F}_M(T)$ is an F-cotilting Λ-module such that*

$$\operatorname{id}_F \mathcal{F}_M(\Gamma) \leq \max\{\operatorname{id}_\Gamma T, 2\} \qquad \text{and} \qquad \operatorname{End}_\Lambda(\mathcal{F}_M(\Gamma)) \simeq \Gamma.$$

If, in addition, M is injective, then

$$\operatorname{id}_F \mathcal{F}_M(\Gamma) \leq \max\{\operatorname{id}_\Gamma T - 2, 0\}.$$

Proposition 8.2 (Iyama/Solberg)**.** *Let Γ be a Gorenstein algebra such that*

$$\operatorname{dom.dim}\Gamma \geq n+1 \geq \operatorname{id}_\Gamma \Gamma.$$

If $M := I(\Gamma)$ and $\Lambda := \operatorname{End}_\Gamma(M)$, then $\mathcal{F}_M(\Gamma)$ is a generator-cogenerator of $\operatorname{mod}\Lambda$ such that $\operatorname{End}_\Lambda(\mathcal{F}_M(\Gamma)) \simeq \Gamma$ and

$$\operatorname{Ext}^i_\Lambda(\mathcal{F}_M(\Gamma), \mathcal{F}_M(\Gamma)) = 0$$

for each $i \in [1, n-1]$. Moreover,

$$\tau_n(\mathcal{F}_M(\Gamma)), \tau_n^-(\mathcal{F}_M(\Gamma)) \in \operatorname{add}\mathcal{F}_M(\Gamma).$$

Proof. Observe that $\Lambda = \mathcal{F}_M(M)$ is a direct summand of $\mathcal{F}_M(\Gamma)$, hence $\mathcal{F}_M(\Gamma)$ is a generator of $\operatorname{mod}\Lambda$. Next, M is a direct summands of $\mathrm{D}(\Gamma)$, hence $\Lambda = \operatorname{Hom}_\Gamma(M, M)$ is a direct summand of $\operatorname{Hom}_\Gamma(M, \mathrm{D}(\Gamma)) \simeq \mathcal{F}_{\mathrm{D}(M)}(\Gamma)$. This implies that $\mathrm{D}(\Lambda)$ is a direct summand of $\mathrm{D}(\mathcal{F}_{\mathrm{D}(M)}(\Gamma)) \simeq \mathcal{F}_M(\Gamma)$, hence $\mathcal{F}_M(\Gamma)$ is a cogenerator of $\operatorname{mod}\Lambda$.

Next, $\operatorname{End}_\Lambda(\mathcal{F}_M(\Gamma)) \simeq \Gamma$ due to the previous theorem, and

$$\operatorname{Ext}^i_\Lambda(\mathcal{F}_M(\Gamma), \mathcal{F}_M(\Gamma)) = 0$$

for each $i \in [1, n-1]$ due to a result of Müller.

Finally, let $F := F_{\mathcal{F}_M(\Gamma)}$. The previous theorem implies that $\mathcal{F}_M(\Gamma)$ is an F-cotilting module with

$$\operatorname{id}_F \mathcal{F}_M(\Gamma) \leq \max\{\operatorname{id}_\Gamma \Gamma - 2, 0\} \leq n - 1.$$

Now, if

$$0 \to \mathcal{F}_M(\Gamma) \to I^0 \to I^1 \to I^2 \to \cdots$$

is the minimal injective resolution of $\mathcal{F}_M(\Gamma)$, then the sequence

$$0 \to \mathcal{F}_M(\Gamma) \to I^0 \to I^1 \to \cdots \to I^{n-2} \to \Omega_\Lambda^{-(n-1)}\mathcal{F}_M(\Gamma) \to 0$$

is F-exact. Consequently,

$$\Omega_\Lambda^{-(n-1)}(\mathcal{F}_M(\Gamma)) \in \mathcal{I}(F) = \operatorname{add}(\mathrm{D}(\Lambda) \oplus \mathrm{D}\operatorname{Tr}(\mathcal{F}_M(\Gamma))),$$

since $\operatorname{id}_F \mathcal{F}_M(\Gamma) \leq n - 1$. Thus

$$\tau_n^-(\mathcal{F}_M(\Gamma)) = \operatorname{Tr}\mathrm{D}\,\Omega_\Lambda^{-(n-1)}(\mathcal{F}_M(\Gamma)) \in \operatorname{add}\mathcal{F}_M(\Gamma),$$

and this finishes the proof. □

An additive subcategory $\mathcal{D}$ of the category of modules over an algebra Λ is called n-precluster tilting if $\Lambda \oplus \mathrm{D}(\Lambda) \in \mathcal{D}$, $\tau_n(\mathcal{D}), \tau_n^-(\mathcal{D}) \subseteq \mathcal{D}$, $\operatorname{Ext}^i_\Lambda(\mathcal{D}, \mathcal{D}) = 0$ for each $i \in [1, n-1]$, and $\mathcal{D}$ is functorially finite. A module M over an algebra Λ is called an n-precluster tilting if $\operatorname{add} M$ is an n-precluster tilting subcategory of $\operatorname{mod}\Lambda$. If Γ is a Gorenstein algebra such that

$$\operatorname{dom.dim}\Gamma \geq n + 1 \geq \operatorname{id}_\Gamma \Gamma,$$

$M := I(\Gamma)$ and $\Lambda := \operatorname{End}_\Gamma(M)$, then $\mathcal{F}_M(\Gamma)$ is an n-precluster tilting Λ-module. Every n-cluster tilting subcategory is an n-precluster tilting subcategory. Finally, if Λ is a $\mathrm{D}\operatorname{Tr}$-selfinjective algebra, $\mathrm{D}\operatorname{Tr}(M) \simeq M$ for a Λ-module M, and

$$T := M \oplus \bigoplus_{n \in \mathbb{N}} (\operatorname{Tr}\mathrm{D})^i(\Lambda),$$

then T is an n-precluster tilting Λ-module.

Proposition 8.3 (Iyama/Solberg). *Let M be an n-precluster tilting module over an algebra Λ. If $\Gamma := \operatorname{End}_\Lambda(M)$, then Γ is a Gorenstein algebra such that*

$$\operatorname{dom.dim}\Gamma \geq n+1 \geq \operatorname{id}_\Gamma \Gamma.$$

Proposition 8.4 (Iyama/Solberg). *Let Λ be an algebra and $\mathcal{M}$ the additive subcategory of $\operatorname{mod}\Lambda$ generated by the modules $\tau_n^{-l}(\Lambda)$, $\tau_n^l(\mathrm{D}(\Lambda))$, $l \in \mathbb{N}$. There exists an n-precluster tilting Λ-module if and only if $\mathcal{M}$ is of finite type and $\operatorname{Ext}^i_\Lambda(\mathcal{M},\mathcal{M}) = 0$ for each $i \in [1, n-1]$.*

It is known that if $\mathcal{D}$ is an n-precluster tilting subcategory of the category of modules over an algebra Λ, then $\mathcal{D}^{\perp_n} = {}^{\perp_n}\mathcal{D}$ and $\mathcal{D}^{\perp_n}$ has n-fold almost split sequences. Moreover, $\mathcal{D}^{\perp_n}$ is a Frobenius category and its stable category is a triangulated category with the Serre functor given by $\Sigma^n\tau_n$.