TANGENT SPACES, MASSEY PRODUCTS, AND APPLICATIONS TO REPRESENTATION THEORY

BASED ON THE TALK BY ANDREW HUBERY

Throughout the presentation K is an algebraically closed field.

Let X be a K-scheme X. For $x \in X(K)$ and $n \in \mathbb{N}_+$ we denote by $T_x^{(n)} X$ the subspace of $T_x X$ consisting of $\xi \in T_x X$ such that there exists $\xi' \in X(K[t]/t^{n+1})$ with $X(\pi)(\xi') = \xi$, where $\pi : K[t]/t^{n+1} \to K[t]/t^2$ is the canonical projection. One shows that

$$\bigcap_{n \in \mathbb{N}_+} T_x^{(n)} X = T_x X_{\mathrm{red}}$$

for generic $x \in X(K)$.

Lemma. *Let A be a finitely generated commutative K-algebra, $n \in \mathbb{N}$, and $\alpha_0, \ldots, \alpha_n \in \operatorname{Hom}_K(A, K)$. If*

$$\xi : A \to K[t]/t^{n+1} \qquad \text{and} \qquad M : A \to \mathbb{M}_{n+1}(K)$$

are given by

$$\xi := \sum_{i \in [0,n]} \alpha_i \cdot t^i$$

and

$$T_{i,j} := \begin{cases} \alpha_{j-i} & j \geq i, \\ 0 & \text{otherwise}, \end{cases} \qquad (i, j \in [1, n+1]),$$

then $\xi \in \operatorname{Hom}_{K\text{-alg}}(A, K[t]/t^{n+1})$ if and only if $M \in \operatorname{mod}_A^{n+1}(K)$.

We remark that the above lemma is a reminiscent of the Massey product.

Now we apply the above construction to schemes of modules. For the rest of the presentation Λ is a finitely generated K-algebra and $d \in \mathbb{N}$.

For $M \in \operatorname{mod}_\Lambda^d(K)$ we put

$$\mathbb{Z}_M := \{ Z \in \operatorname{Hom}_K(A, \mathbb{M}_d(K)) : [\begin{smallmatrix} M & Z \\ 0 & M \end{smallmatrix}] \in \operatorname{mod}_\Lambda^{2 \cdot d}(K) \}.$$

Then $\mathbb{Z}_M = T_M \operatorname{mod}_\Lambda^d$ for each $M \in \operatorname{mod}_\Lambda^d(K)$. For $M \in \operatorname{mod}_\Lambda^d(K)$ and $n \in \mathbb{N}_+$ we denote by $\mathbb{Z}_M^{(n)}$ the set of all $Z \in \mathbb{Z}_M$ such that there

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exist $Z_2, \dots, Z_n \in \mathrm{Hom}_K(A, \mathbb{M}_d(K))$ with $N \in \mathrm{mod}_\Lambda^{(n+1)\cdot d}(K)$, where $N \in \mathrm{Hom}_K(A, \mathbb{M}_{n+1}(\mathbb{M}_d(K))$ is given by

$$N_{i,j} := \begin{cases} Z_{j-i} & j \geq i, \\ 0 & \text{otherwise,} \end{cases} \qquad (i, j \in [1, n+1]),$$

and $Z_0 := M$ and $Z_1 := Z$. Then

$$T_M(\mathrm{mod}_\Lambda^d)_{\mathrm{red}} = \bigcap_{n \in \mathbb{N}_+} \mathbb{Z}_M^{(n)}$$

for generic $M \in \mathrm{mod}_\Lambda^{(n)}$. Using this we show that if $M \in \mathrm{mod}_\Lambda^d(K)$, then $\overline{\mathcal{O}}_M$ is an irreducible component of $\mathrm{mod}_\Lambda^d(K)$ if and only if

$$\bigcap_{n \in \mathbb{N}_+} \mathbb{Z}_M^{(n)} = \mathbb{B}_M := \{h \cdot M - M \cdot h : h \in \mathbb{M}_d(K)\}.$$

The above construction can be also used in the proof of the following.

Theorem (Crawley-Boevey/Schröer)**.** *Let C_1 and C_2 be irreducible components of $\mathrm{mod}_\Lambda^{d_1}(K)$ and $\mathrm{mod}_\Lambda^{d_2}(K)$ for $d_1, d_2 \in \mathbb{N}$ such that $d_1 + d_2 = d$. Then $\overline{C_1 \oplus C_2}$ is an irreducible component of $\mathrm{mod}_\Lambda^d(K)$ if and only if*

$$e := \min\{\dim_K(\mathrm{Ext}_\Lambda^1(M_1, M_2) \oplus \mathrm{Ext}_\Lambda^1(M_2, M_1)) : M_1 \in C_1 \text{ and } M_2 \in C_2\} = 0.$$

Proof. If $\overline{C_1 \oplus C_2}$ is an irreducible component of $\mathrm{mod}_\Lambda^d(K)$, then $e = 0$, since $\mathcal{O}_{N_1 \oplus N_2} \subseteq \overline{\mathcal{O}}_M$ for each exact sequence of the form

$$0 \to N_1 \to M \to N_2 \to 0.$$

Now assume that $e = 0$. We treat C_1 and C_2 as schemes with the structures induced by the primary ideals in the primary decomposition of 0 corresponding to the prime ideals of C_1 and C_2, respectively. Let $\Phi : \mathrm{GL}_d \times C_1 \times C_2 \to \mathrm{mod}_\Lambda^d$ be given by

$$\Phi(g, M_1, M_2) := g * (M_1 \oplus M_2) \qquad (g \in \mathrm{GL}_d(K),\ M_1 \in C_1(K),\ M_2 \in C_2(K)).$$

Then

$$\mathrm{Coker}\, d\Phi_{(1, M_1, M_2)} = \mathrm{Ext}_\Lambda^1(M_1, M_2) \oplus \mathrm{Ext}_\Lambda^1(M_2, M_1)$$

for generic $M_1 \in C_1(K)$ and $M_2 \in C_2(K)$, hence $d\Phi_{(1,M_1,M_2)}$ is generically surjective, since $e = 0$. Next, using the assumption $e = 0$ and the above construction, we show that $d(\Phi_{\mathrm{red}})_{(1,M_1,M_2)}$ is generically surjective, which implies that $\overline{C_1 \oplus C_2} = \overline{\mathrm{Im}\, \Phi}$ is an irreducible component of $\mathrm{mod}_\Lambda^d(K)$. $\square$