THE BATALIN-VILKOVISKY STRUCTURE OVER THE HOCHSCHILD COHOMOLOGY RING OF A SYMMETRIC ALGEBRA

BASED ON THE TALK BY GUODONG ZHOU

The presentation is bases on a joint work with Jue Le. Throughout the presentation k is a field and all algebras are finite dimensional ones.

1. Inroduction

Throughout this section A is an algebra.

For $n \in \mathbb{N}$ we put

$$C_n(A) := A^{\otimes(n+1)}.$$

Next, for $n \in \mathbb{N}_+$ we define $d : C_n(A) \to C_{n-1}(A)$ by

$$d(a_0 \otimes \cdots \otimes a_n) := \sum_{i \in [0,n-1]} (-1)^i \cdot a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n + (-1)^n \cdot a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \qquad (a_0, \ldots, a_n \in A).$$

Finally, for $n \in \mathbb{N}$ we define the n-th Hochschild homology group $\mathrm{HH}_n(A)$ by

$$\mathrm{HH}_n(A) := H_n(C_*(A)).$$

One shows that $\mathrm{HH}_n(A) = \mathrm{Tor}_n^{A^e}(A, A)$ for each $n \in \mathbb{N}$. We have the Connes B-operator $B : C_n(A) \to C_{n+1}(A)$ $(n \in \mathbb{N})$ defined by

$$B(a_0 \otimes \cdots \otimes a_n) := \sum_{i \in [0,n]} (-1)^{i \cdot n} \cdot 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-1} - \sum_{i \in [0,n]} (-1)^{(i-1) \cdot n} \cdot a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-2} \qquad (a_0, \ldots, a_n \in A),$$

which induces the map $B : \mathrm{HH}_n(A) \to \mathrm{HH}_{n+1}(A)$ $(n \in \mathbb{N})$ such that $B^2 = 0$.

Analogously, for $n \in \mathbb{N}$ we put

$$C^n(A) := \mathrm{Hom}_k(A^{\otimes n}, A),$$

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and for $n \in \mathbb{N}_+$ we define $d : C^n(A) \to C^{n+1}(A)$ by

$$(df)(a_0 \otimes \cdots \otimes a_n) := a_0 \cdot f(a_1 \otimes \cdots \otimes a_n)$$
$$+ \sum_{i \in [0,n-1]} (-1)^{i+1} \cdot f(a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i \cdot a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n)$$
$$+ (-1)^{n+1} \cdot f(a_0 \otimes \cdots \otimes a_{n-1}) \cdot a_n$$
$$(f \in C^n(A),\ a_0, \ldots, a_n \in A).$$

If $n \in \mathbb{N}$, then by the n-the Hochschild cohomology group $\mathrm{HH}^n(A)$ of A we mean $H^n(C^*(A))$. One shows that $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$ for each $n \in \mathbb{N}$. If $n, m \in \mathbb{N}$, then the following operation $C^n(A) \times C^m(A) \to C^{n+m}(A)$,

$$(f, g) \mapsto (a_1 \otimes \cdots \otimes a_{n+m} \mapsto f(a_1 \otimes \cdots \otimes a_n) \cdot g(a_{n+1} \otimes \cdots \otimes a_{n+m}))$$
$$(f \in C^n(A),\ g \in C^m(A),\ a_1, \ldots, a_{n+m} \in A),$$

induces the cup product

$$\cup : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \to \mathrm{HH}^{n+m}(A).$$

We also have the Gerstenhaber bracket

$$[-, -] : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \to \mathrm{HH}^{n+m-1}(A).$$

Theorem (Gerstenhaber)**.** $\mathrm{HH}^*(A)$ *together with* $\cup$ *and* $[-, -]$ *is a Gerstenhaber algebra, i.e.*

(1) $\mathrm{HH}^*(A)$ *together with* $\cup$ *is a graded commutative algebra, i.e.*

$$f \cup g = (-1)^{|f| \cdot |g|} \cdot g \cup f$$

for all $f, g \in \mathrm{HH}^*(A)$*,*

(2) $\mathrm{HH}^*(A)$ *together with* $[-, -]$ *is a graded Lie algebra of degree* -1*, i.e.*

$$[f, g] = (-1)^{(|f|-1)\cdot(|g|-1)} \cdot [g, f]$$

for all $f, g \in \mathrm{HH}^*(A)$*, and*

$$(-1)^{(|f|-1)\cdot(|h|-1)} \cdot [[f, g], h] + (-1)^{(|g|-1)\cdot(|f|-1)} \cdot [[g, h], f]$$
$$+ (-1)^{(|h|-1)\cdot(|g|-1)} \cdot [[h, f], g] = 0$$

for all $f, g, h \in \mathrm{HH}^*(A)$*,*

(3) *and the Poison rule is satisfied, i.e.*

$$[f \cup g, h] = [f, h] \cup g + (-1)^{|f|\cdot(|h|-1)} \cdot f \cup [g, h]$$

for all $f, g, h \in \mathrm{HH}^*(A)$*.*

2. The Hochschild cohomology for a symmetric algebra

Recall that an algebra A is called symmetric if and only if the A^e-modules A and DA are isomorphic, and if and only if there exists a symmetric bilinear nondegenerate form $\langle -,- \rangle : A \times A \to k$ which is associative (i.e. $\langle a \cdot b, c\rangle = \langle a, b \cdot c\rangle$ for all $a, b, c \in A$). If A is a symmetric algebra, then $\mathrm{HH}^n(A) \simeq D(\mathrm{HH}^n(A))$ for each $n \in \mathbb{N}$, and via this isomorphism the Connes B-operator induces the map $\Delta : \mathrm{HH}^n(A) \to \mathrm{HH}^{n-1}(A)$ $(n \in \mathbb{N})$.

Theorem (Tadler)**.** *If A is a symmetric algebra, then $\mathrm{HH}^*(A)$ together with $\cup$, $[-,-]$, and Δ, is a Batalin-Vilkovisky algebra (shortly, BV-algebra), i.e.*

(1) *$\mathrm{HH}^*(A)$ together with $\cup$ and $[-,-]$ is a Gerstenhaber algebra,*
(2) *$\Delta^2 = 0$,*
(3) *and*

$$[f, g] = -(-1)^{(|f|-1)\cdot|g|} \cdot (\Delta(f \cup g) - \Delta f \cup g - (-1)^{|f|} \cdot f \cup \Delta g)$$

for all $f, g \in \mathrm{HH}^*(A)$.

3. Operations on symmetric algebras

Let A and A' be symmetric algebras via forms

$$\langle -,- \rangle : A \times A \to k \qquad \text{and} \qquad \langle -,- \rangle' : A' \times A' \to k,$$

respectively. Then $A \otimes A'$ is a symmetric algebra via the form

$$(a \otimes a', b \otimes b') \mapsto \langle a, b\rangle \cdot \langle a', b'\rangle' \qquad (a, b \in A,\ a', b' \in A').$$

Moreover, there exists an isomorphism

$$\mathrm{HH}^*(A \otimes A') \to \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(A')$$

of BV-algebras, where for BV-algebras H^* and L^* we define $H^* \otimes L^*$ in the following way. First, we put

$$(H^* \otimes L^*)^n := \bigoplus_{s \in [0,n]} H^s \otimes L^{n-s} \qquad (n \in \mathbb{N}).$$

Next, we define

$$(f \otimes f') \cup (g \otimes g') := (-1)^{|f'|\cdot|g|}(f \cup g) \otimes (f' \cup g') \qquad (f, g \in H^*,\ f', g' \in L^*),$$

and

$$[f \otimes f', g \otimes g'] := (-1)^{(|f|+|f'|-1)\cdot|g'|} \cdot [f, g] \otimes (f' \cup g') + (-1)^{|f|\cdot(|g|+|g'|-1)}(f \cup g) \cdot [f', g'] \qquad (f, g \in H^*,\ f', g' \in L^*).$$

Finally, we put

$$\Delta(f \otimes f') := \Delta f \otimes f' + (-1)^{|f|} \cdot f \otimes \Delta f' \qquad (f \in H^*,\ f' \in L^*).$$

This is an open question, if there exists an isomorphism

$$\mathrm{HH}^*(A \otimes A') \to \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(A')$$

of Gerstenhaber algebras for arbitrary algebras A and A'.

Again, let A and A' be symmetric algebras via forms

$$\langle -, - \rangle : A \times A \to k \qquad \text{and} \qquad \langle -, - \rangle' : A' \times A' \to k,$$

respectively. Then $A \times A'$ is a symmetric algebra via the form

$$((a, a'), (b, b')) \mapsto \langle a, b \rangle + \langle a', b' \rangle' \qquad (a, b \in A,\ a', b' \in A'),$$

and there is an isomorphism

$$\mathrm{HH}^*(A \times A') \to \mathrm{HH}^*(A) \times \mathrm{HH}^*(A')$$

of BV-algebras.

Finally, let k' be a field extensions of k. If A is a symmetric algebra via a form $\langle -, - \rangle \to k$, then $A \otimes k'$ is a symmetric algebra via the form

$$(a \otimes \lambda, b \otimes \mu) \mapsto \langle a, b \rangle \otimes (\lambda \cdot \mu) \qquad (a, b \in A,\ \lambda, \mu \in k),$$

and there is an isomorphism

$$\mathrm{HH}^*(A \otimes k') \to \mathrm{HH}^*(A) \otimes k'$$

of BV-algebras.

4. Example

Let $A := k[X]/X^n$ for some $n \in \mathbb{N}_+$. Due to results of Holm, Suárez-Álvarez, and Yang,

$$\mathrm{HH}^*(A) = \begin{cases} k[x, y, z]/(x^n, y^2) \\ \qquad p \mid n \text{ and either } p \neq 2 \text{ or } 4 \mid n, \\ k[x, y, z]/(x^n, y^2 - x^{n-2} \cdot z) \\ \qquad p \mid n,\ p = 2, \text{ and } 4 \nmid n, \\ k[x, y, z]/(x^n, x^{n-1} \cdot z, y \cdot x^{n-1}, y^2) & p \nmid n, \end{cases}$$

where $|x| = 0$, $|y| = 1$, and $|z| = 2$. Moreover, if $r \in [0, n-1]$ and $t \in \mathbb{N}$, then $\Delta(x^r \cdot z^t) = 0$ and

$$\Delta(x^r \cdot y \cdot z^t) = \begin{cases} ((t+1) - n \cdot (r+1)) \cdot x^r \cdot z^t & p \nmid n, \\ r \cdot x^{r-1} \cdot z^t & p \mid n. \end{cases}$$