CLUSTER TILTED ALGEBRAS

BASED ON THE TALKS BY ASLAK BAKKE BUAN

1. Quiver mutations

Let C be an $n \times n$-matrix with integer coefficients such that $C(i,j) \geq 0$ for all $i, j \in [1, n]$ and $C(i,j) \cdot C(j,i) = 0$ for all $i, j \in [1, n]$ (in particular, $C(i,i) = 0$ for all $i \in [1, n]$). Following [11] by a mutation of C at $k \in [1, n]$ we mean the $n \times n$-matrix $\mu_k C$ defined by

$$\mu_k C(i,j) := \begin{cases} C(j,i) & \text{if } i = k \text{ or } j = k, \\ \max(0, C(i,j) - C(j,i) \\ \qquad + C(i,k) \cdot C(k,j) - C(j,k) \cdot C(k,i)) & \\ & \text{otherwise,} \end{cases}$$

$$(i, j \in [1, n]).$$

One can easily check that $\mu_k C$ has the same properties as C, i.e. $\mu_k C(i,j) \geq 0$ for all $i, j \in [1, n]$ and $\mu_k C(i,j) \cdot \mu_k C(j,i) = 0$ for all $i, j \in [1, n]$. Moreover, $\mu_k^2 C = C$.

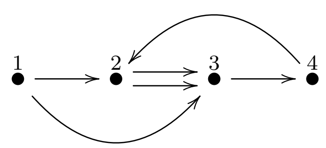
With a matrix C as above we can associate a quiver Q such that $Q_0 = [1, n]$ and

$$\#\{\alpha \in Q_1 : s\alpha = i \text{ and } t\alpha = j\} = C(i,j)$$

for all $i, j \in [1, n]$. The quiver Q is uniquely determined by C up to an isomorphism fixing vertices. Moreover, Q has no loops and no oriented 2-cycles. If $k \in [1, n]$ and Q' is the quiver associated with $\mu_k C$, then we write $Q' = \mu_k Q$ and call Q' the mutation of Q at k. Observe that Q' is obtained from Q in the following way:

(1) if i and j are vertices of Q, then we add an arrow from i to j for every path from i to j of length 2 going through k,
(2) we reverse all arrows which start or terminate in k,
(3) we remove oriented 2-cycles until no oriented 2-cycles are left.

For example, if Q is the quiver



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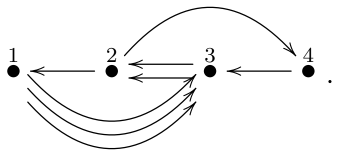
then $\mu_2 Q$ equals

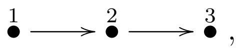


Observe that the mutation at k is the reflection at k provided k is either a sink or a source.

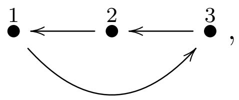
Let Q be an acyclic quiver and denote by H its path algebra (over a fixed algebraically closed field). For a sink k in Q we define the tilting module T by

$$T := H/P_k \amalg \tau^{-1} P_k.$$

Then $\mathrm{End}_H(T)^{\mathrm{op}}$ is (isomorphic to) the path algebra of the mutation of Q at k. Note however that we cannot expect such a result for general mutations. Indeed, if Q is the quiver



then $\mu_2 Q$ equals



hence there is no (iterated) tilted algebra whose Gabriel quiver equals $\mu_2 Q$. One of the aims of introducing cluster categories was to find a similar interpretation for arbitrary mutations.

2. Cluster categories and tilting

Let Q be an acyclic quiver, denote by H its path category and by $\mathcal{D}_H$ the derived category of H. It is a triangulated Krull–Schmidt category with the suspension functor given by the shift [1] of complexes. Moreover, it has AR-triangles, thus in particular, we have the AR-translation τ. If X is an indecomposable object in $\mathcal{D}_H$, then there exists an indecomposable H-module M such that $X \simeq M[i]$ for some $i \in \mathbb{Z}$.

Let $F := \tau^{-1} \circ [1]$. We put $\mathcal{C} = \mathcal{C}_H := \mathcal{D}_H / F$, i.e. $\mathcal{C}_H$ has the same objects as $\mathcal{D}_H$ and

$$\mathrm{Hom}_{\mathcal{C}_H}(X, Y) := \coprod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}_H}(X, F^i Y)$$

for objects X and Y in $\mathcal{C}$. Then $\mathcal{C}$ is again a triangulated Krull–Schmidt category such that the canonical functor $\mathcal{D}_H \to \mathcal{C}$ is a triangle functor [13]. Moreover, $\mathcal{C}$ has AR-triangles and each indecomposable object in $\mathcal{C}$ is isomorphic either to M for an indecomposable H-module M or to $P[1]$ for an indecomposable projective H-module P.

If $T = \bigoplus_{i\in[1,n]} X_i$ for indecomposable objects $X_1, \ldots, X_n \in \mathcal{C}$, then we put $\delta(T) := n$. Moreover, if $X_i \not\simeq X_j$ for all $i, j \in [1, n]$, $i \neq j$, then T is called basic. An object T in $\mathcal{C}$ is called tilting if T is basic, $\mathrm{Ext}^1_{\mathcal{C}}(T, T) = 0$, and $\delta(T) = |Q_0|$.

Lemma ([5])**.**

(1) *If T is a tilting H-module, then T is a tilting object in $\mathcal{C}$.*
(2) *If T is a tilting object in $\mathcal{C}$, then there exists a hereditary algebra H', a triangle equivalence $F : \mathcal{D}_{H'} \to \mathcal{D}_H$, and a tilting H'-module T', such that $T \simeq FT'$.*

An object T of $\mathcal{C}$ is called almost tilting if T is basic, $\mathrm{Ext}^1_{\mathcal{C}}(T, T) = 0$, and $\delta(T) = |Q_0| - 1$. If T is an almost tilting object in $\mathcal{C}$, then M is called a complement of T, if $T \amalg M$ is a tilting object. Obviously, if M is a complement of an almost tilting object T, then M is indecomposable.

Proposition ([5])**.** *Let T be an almost tilting object in $\mathcal{C}$. Then there exist exactly two* (*up to isomorphism*) *complements of T. Moreover, if M and M^* are the complements of T, then there exist essentially unique triangles*

$$M^* \xrightarrow{f} B \xrightarrow{g} M \to M^*[1] \qquad \text{and} \qquad M \xrightarrow{f'} B' \xrightarrow{g'} M^* \to M[1]$$

in $\mathcal{C}$, such that f and f' are minimal left $\mathrm{add}\, T$*-approximations, while g and g' are minimal right* $\mathrm{add}\, T$*-approximations.*

For an algebra Λ we denote by Q_Λ its Gabriel quiver. Recall that there exists a bijection between the isomorphism classes of the indecomposable projective Λ-modules and the vertices of Q_Λ. In particular, if T is a tilting object in $\mathcal{C}$, then there exists a bijection between the isomorphism classes of the indecomposable direct summands of T and the vertices of $Q_{\mathrm{End}_{\mathcal{C}}(T)}$.

Theorem (Buan/Marsh/Reiten [8])**.** *Let M and M^* be the complements of an almost tilting module in $\mathcal{C}_H$. Then*

$$Q_{\mathrm{End}(T \amalg M^*)^{\mathrm{op}}} = \mu_k Q_{\mathrm{End}(T \amalg M)^{\mathrm{op}}},$$

where k is the vertex of $Q_{\mathrm{End}(T \amalg M)^{\mathrm{op}}}$ corresponding to $[M]$.

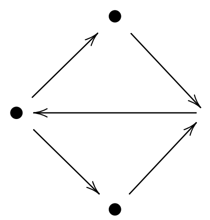
By the tilting graph of $\mathcal{C}$ we mean the graph whose vertices are the isomorphism classes of the tilting objects in $\mathcal{C}$ and there is an edge $[T'] - [T'']$ if and only if there exist an almost tilting object T and indecomposable objects M and M^* such that $T' \simeq T \amalg M$ and $T'' \simeq T \amalg M^*$.

Proposition ([5])**.** *The tilting graph is connected.*

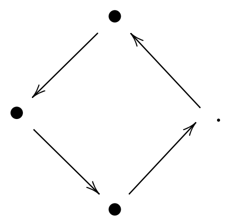
We say that quivers Q' and Q'' without loops and oriented 2-cycles are mutation equivalent if there exists a sequence $k_1, \ldots, k_n$ of vertices of Q'' such that

$$Q' = \mu_{k_1} \cdots \mu_{k_n} Q''.$$

By the mutation class of a quiver Q' without loops and oriented 2-cycles we mean the set of the isomorphism classes of the quivers, which are mutation equivalent to Q'. For example, the mutation class of a Dynkin quiver of type $\mathbb{D}_4$ consists of the isomorphism classes of the Dynkin quivers of type $\mathbb{D}_4$ and the isomorphism classes of the following quivers



and



By a cluster tilted algebra of type H we mean every algebra of the form $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, where T is a tilting object in $\mathcal{C}$.

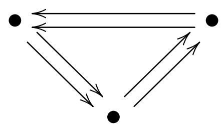
Theorem ([8]). *The mutation class of Q consists of the isomorphism classes of the Gabriel quivers of the cluster tilted algebras of type H.*

Theorem (Buan/Reiten [9]). *The mutation class of Q is finite if and only if $|Q_0| = 2$ or Q is Dynkin or Euclidean.*

Proof. In order to prove that the mutation class of Q is finite if Q is Euclidean we use the following facts:

- every tilting module over an Euclidean quiver has a non-regular direct summand,
- if T is a preprojective module over an Euclidean quiver, then there are only finitely many isomorphism classes of the indecomposable modules X such that $\mathrm{Ext}^1(T \amalg X, T \amalg X) = 0$,
- if T is a tilting object in $\mathcal{C}$, then $\mathrm{End}_{\mathcal{C}}(T) \simeq \mathrm{End}_{\mathcal{C}}(\tau T)$. □

Recall that Q is an acyclic quiver in the above theorem. Note that



is a quiver, which is mutation equivalent neither to a Dynkin nor to a Euclidean quiver, but whose mutation class is finite – in fact, its mutation class consists of its isomorphism class alone. There is a generalization of the above theorem due to Felikson, Shapiro and Tumarkin [10] describing the quivers without loops and oriented 2-cycles having a finite mutation class.

A triangulated category $\mathcal{T}$ is called 2-Calabi–Yau if

$$\mathrm{Ext}^1_{\mathcal{T}}(A, B) \simeq D\,\mathrm{Ext}^1_{\mathcal{T}}(B, A)$$

for all objects A and B in $\mathcal{T}$. The Auslander–Reiten formula implies that the cluster categories are examples of 2-Calabi–Yau categories. Other examples of 2-Calabi–Yau triangulated Hom-finite categories are the stable module categories for preprojective algebras studied by Geiss, Leclerc and Schröer [16], and the cluster categories for quivers with potentials introduced by Amiot [1] and Plamondon [17].

The following theorem describes the module category over a cluster tilted algebra.

Theorem (Buan/Marsh/Reiten [7]). *If T is a tilting object in $\mathcal{C}$, then the functor*

$$\operatorname{Hom}_{\mathcal{C}}(T,-) : \mathcal{C} \to \operatorname{mod} \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$$

is full and dense, and its kernel consists of the morphisms which factor through $\operatorname{add} T[1]$.

We have the following comparison of the module categories of two adjacent cluster tilted algebras.

Theorem (Buan/Marsh/Reiten [7]). *Let M and M^* be the complements of an almost tilting object T in $\mathcal{C}$. If*

$$S_M := \operatorname{top} \operatorname{Hom}_{\mathcal{C}}(T \amalg M, M)$$

and

$$S_{M^*} := \operatorname{top} \operatorname{Hom}_{\mathcal{C}}(T \amalg M^*, M^*),$$

then we have an equivalence

$$\operatorname{mod} \operatorname{End}_{\mathcal{C}}(T \amalg M)^{\mathrm{op}} / \operatorname{add} S_M \simeq \operatorname{mod} \operatorname{End}_{\mathcal{C}}(T \amalg M^*)^{\mathrm{op}} / \operatorname{add} S_{M^*}.$$

The next theorem presents basic homological properties of the cluster tilted algebras.

Theorem (Keller/Reiten [14]). *If Γ is a cluster tilted algebra, then*

$$\operatorname{id}_{\Gamma} \Gamma \leq 1.$$

In particular,

$$\operatorname{gldim} \Gamma \in \{0, 1, \infty\}.$$

Finally, we may describe the cluster tilted algebras in an alternative way using the following result.

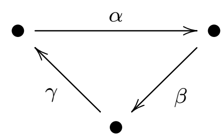
Theorem (Assem/Brüstle/Schiffler [2]). *If T is a tilting H-module, then*

$$\operatorname{Ext}_{\mathcal{C}}(T)^{\mathrm{op}} \simeq \Lambda \ltimes \operatorname{Ext}^2_{\Lambda}(D\Lambda, \Lambda),$$

where $\Lambda := \operatorname{End}_H(T)^{\mathrm{op}}$.

3. Quivers and relations for cluster tilted algebras

By a potential in a quiver Q we mean a linear combination of oriented cycles in Q. Given a quiver Q and a potential w we define the algebra $J_{Q,w}$ as the quotient of the path algebra of Q by the ideal generated by the relations $\frac{\partial w}{\partial \alpha}$, $\alpha \in Q_1$. For example, if Q is the quiver



and $w = \gamma\beta\alpha$, then $J_{Q,w}$ is the path algebra of Q modulo the ideal generated by the relations

$$\beta\alpha, \alpha\gamma, \beta\alpha.$$

Algebras of the above form are called Jacobian algebras.

Theorem (Buan/Iyama/Reiten/Smith, Keller)**.** *If Γ and Γ' are cluster tilted algebras such that $Q_\Gamma = Q'_\Gamma$, then $\Gamma \simeq \Gamma'$. Moreover, every cluster tilted algebra is a Jacobian algebra.*

Buan, Marsh and Reiten described how to find for a cluster tilted algebra Γ of finite representation type a potential w in Q_Γ such that $\Gamma \simeq J_{Q_\Gamma,w}$. This result was generalized by Barot and Trepode to cluster tilted algebras Γ such that there are no double arrows in Q_Γ.

4. From triangulated categories to module categories via localizations

Let $\mathcal{C}$ be a triangulated Hom-finite Krull–Schmidt category with the suspension functor Σ. König and Zhu [15], and, independently, Iyama and Yoshino [12], proved, that if $\operatorname{Ext}^1_{\mathcal{C}}(T,T) = 0$ and

$$\operatorname{add} T = \{X \in \mathcal{C} : \operatorname{Ext}^1_{\mathcal{C}}(T,X) = 0\},$$

then the functor

$$\operatorname{Hom}_{\mathcal{C}}(T,-) : \mathcal{C} \to \operatorname{mod} \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$$

is full and dense, and its kernel consists of the morphisms which factor through $\operatorname{add} \Sigma T$. Our aim is to study the functor

$$\operatorname{Hom}_{\mathcal{C}}(T,-) : \mathcal{C} \to \operatorname{mod} \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$$

for $T \in \mathcal{C}$ such that $\operatorname{Ext}^1_{\mathcal{C}}(T,T) = 0$.

Let $\mathcal{X}_T$ be the class of the objects X in $\mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(T,X) = 0$. Let $\mathcal{S}$ be the class of the maps $f : X \to Y$ such that g and h factor through $\mathcal{X}_T$, where

$$\Sigma^{-1} Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} Z$$

is a triangle.

Lemma ([4]). *If f is a morphism in $\mathcal{C}$, then $\operatorname{Hom}_{\mathcal{C}}(T, f)$ is an isomorphism if and only if $f \in \mathcal{S}$.*

Let $L_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ be the Gabriel–Zismas localization of $\mathcal{C}$ with respect to $\mathcal{S}$. More precisely, the category $\mathcal{C}_{\mathcal{S}}$ has the same objects as $\mathcal{C}$. In order to define the maps in $\mathcal{C}_{\mathcal{S}}$ we first define the graph $\mathcal{G}$ whose vertices are the objects of $\mathcal{C}$ and the arrows are the maps in $\mathcal{C}$ and the arrows $x_s : Y \to X$ for each map $s : X \to Y$ from $\mathcal{S}$. The maps from A to B in $\mathcal{C}_{\mathcal{S}}$ are the equivalence classes of the paths from A to B in $\mathcal{G}$ modulo the equivalence relation generated by the relations

$$x_s \circ s \sim \mathrm{id} \sim s \circ x_s,$$

where $s \in \mathcal{S}$, and

$$f \circ g \sim fg,$$

where f and g are composable maps in $\mathcal{C}$. Finally, $L_{\mathcal{S}}$ is the canonical functor. Then $L_{\mathcal{S}}(s)$ is an isomorphism for each map $s \in \mathcal{S}$ and $L_{\mathcal{S}}$ is universal with respect to this property.

Theorem ([4]). *There exists an equivalence $F : \mathcal{C}_{\mathcal{S}} \to \operatorname{mod} \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ such that*

$$\operatorname{Hom}_{\mathcal{C}}(T, -) = F \circ L_{\mathcal{S}}.$$

Observe that if

$$\operatorname{add} T = \{X \in \mathcal{C} : \operatorname{Ext}^1_{\mathcal{C}}(T, X) = 0\},$$

then there is a natural equivalence $\mathcal{C}/\Sigma T \simeq \mathcal{C}_{\mathcal{S}}$. Note that, in general, there are no left/right fractions for $\mathcal{S}$ in $\mathcal{C}$.

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