ALGEBRAS, MODULES AND CATEGORIES ASSOCIATED WITH ELEMENTS IN COXETER GROUPS

BASED ON THE TALKS BY IDUN REITEN

Throughout the talk k is an algebraically closed field and Q is a quiver without oriented cycles. Moreover, we put $n := |Q_0|$.

1. Coxeter groups

We define a group C, called the Coxeter group associated with Q, in the following way: C has generators s_i, $i \in Q_0$, which are subject to the following relations:

- $s_i^2 = 1$ for each $i \in Q_0$,
- $s_i s_j = s_j s_i$ for all $i, j \in Q_0$ such that there is no arrow between i and j in Q,
- $s_i s_j s_i = s_j s_i s_j$ for all $i, j \in Q_0$ such that there is exactly one arrow between i and j in Q.

For example, if Q is the following quiver

$$\underset{1}{\bullet} \longleftarrow \underset{2}{\bullet} ,$$

then C consists of the following elements

$$1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1 = s_2 s_1 s_2$$

and is isomorphic to S_3. In general, if Q is of type $\mathbb{A}_n$, then C is isomorphic to S_{n+1}. It is known that C is finite if and only if Q is a Dynkin quiver.

For a sequence $\omega = (i_1, \ldots, i_l) \in Q_0^l$, $l \in \mathbb{N}$, we define an element $w(\omega)$ of C by

$$w(\omega) := s_{i_1} \cdots s_{i_l}.$$

If $w \in C$, then we define the length $\ell(w)$ of w by

$$\ell(w) := \min\{l \in \mathbb{N} : \text{there exists } \omega \in Q_0^l \text{ such that } w(\omega) = w\}.$$

If Q is a Dynkin quiver, then there exists a unique element of maximal length in C. A sequence $\omega \in Q_0^l$, $l \in \mathbb{N}$, is said to be reduced if $l = \ell(w(\omega))$.

Let $\omega = (i_1, \ldots, i_l) \in Q_0^l$ and $\omega' = (j_1, \ldots, j_l) \in Q_0^l$, $l \in \mathbb{N}$. If there exists $p \in [1, l-1]$ such that there is no arrow between i_p and i_{p+1}, $i_p = j_{p+1}$, $j_p = i_{p+1}$, and $i_q = j_q$ for all $q \in [1, l]$, $q \neq p, p+1$ (i.e., ω' is obtained from ω by replacing (i_p, i_{p+1}) by (i_{p+1}, i_p)), then $w(\omega) =$

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$w(\omega')$. Similarly, if there exists $p \in [1, l-2]$ such that there is exactly one arrow between i_p and i_{p+1}, $i_p = j_{p+1} = i_{p+2}$, $j_p = i_{p+1} = j_{p+2}$, and $i_q = j_q$ for all $q \in [1, l]$, $q \neq p, p+1, p+2$, then $w(\omega) = w(\omega')$. One can show, that if both ω and ω' are reduced and $w(\omega) = w(\omega')$, then ω' can be obtained from ω by a sequence of the above operations.

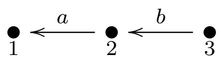
By an admissible ordering of the vertices of Q we mean a bijection $\sigma : [1, n] \to Q_0$ such that there is no arrow from $\sigma(i)$ to $\sigma(j)$ if $i, j \in [1, n]$ and $i < j$. By the Coxeter element we mean $\omega(\sigma(1), \ldots, \sigma(n))$, where σ is an admissible ordering of the vertices of Q. One can show that this definition does not depend on the choice of σ.

2. Algebras associated with elements in Coxeter groups

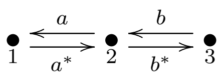
Let $\overline{Q}$ be the double quiver of Q, i.e. $\overline{Q}_0 := Q_0$ and for each arrow $a : x \to y$ in Q we have two arrows $a : x \to y$ and $a^* : y \to x$ in $\overline{Q}$. By Λ we denote the preprojective algebra associated with Q, i.e.

$$\Lambda := k\overline{Q} / \Big\langle \sum_{a \in Q_1} a^* a - a a^* \Big\rangle.$$

For example, if Q is the quiver



then $\overline{Q}$ is the following quiver



and

$$\Lambda = k\overline{Q} / \langle a^* a, b^* b - a a^*, b b^* \rangle.$$

In particular, the indecomposable projective Λ-modules can be visualized as follows:

$$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \quad , \quad \begin{matrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{matrix} \quad , \quad \begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \quad .$$

One shows that Λ is finite dimensional if and only if Q is a Dynkin quiver.

If $i \in Q_0$, then we define an ideal I_i of Λ by

$$I_i := \Lambda(1 - e_i)\Lambda.$$

Next, if $w = w(i_1, \ldots, i_l)$ for reduced $(i_1, \ldots, i_l) \in Q_0^l$, $l \in \mathbb{N}$, then we define an ideal I_w of Λ by

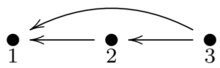
$$I_w := I_{i_1} \cdots I_{i_l}.$$

One can show that I_w does not depend on the choice of $(i_1, \ldots, i_l)$. Finally, for $w \in C$ we define an algebra Λ_w by

$$\Lambda_w := \Lambda / I_w.$$

It is known that Λ_w is finite dimensional for each $w \in C$. Moreover, if Q is Dynkin and w is the element of the longest length in C, then $\Lambda_w \simeq \Lambda$.

There exists a combinatorial rule for describing the indecomposable projective modules in the algebras of the above form, which we illustrate by the following example. Let Q be the quiver



and $w = s_1 s_2 s_3 s_1 s_2$. Then $P_1 := \Lambda e_1$ can be visualized by the following infinite diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 2 & & 3 & & & \\
 & & 3 & & 1 & & 2 & & \\
 & 1 & & 2 & & 3 & & 1 & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}.$$

Now, for

$$P_1/I_2P_1,\ P_1/I_1I_2P_1,\ P_1/I_3I_1I_2,\ P_1/I_2I_3I_1I_2P_1$$

and

$$\Lambda_w e_1 = P_1/I_1I_2I_3I_1I_2P_1$$

we get the diagrams

$$\varnothing \quad , \quad 1 \quad , \quad \begin{array}{cc} 1 & \\ & 3 \end{array} \quad , \quad \begin{array}{cccc} & 1 & & \\ 2 & & 3 & \\ & & & 2 \end{array}$$

and

$$\begin{array}{ccccc}
 & 1 & & & \\
2 & & 3 & & \\
 & 1 & & 2 & \\
 & & & & 1
\end{array},$$

respectively. Similarly, $\Lambda_w e_2$ and $\Lambda_w e_3$ can be visualized by the diagrams

$$\begin{array}{ccccccc}
 & & & & 2 & & \\
 & & & 1 & & 3 & \\
 & & 3 & & 2 & & 1 \\
 & 2 & & 1 & & & \\
1 & & & & & &
\end{array} \quad \text{and} \quad \begin{array}{cccc}
 & 3 & & \\
1 & & 2 & \\
 & & & 1
\end{array},$$

respectively.

3. Cluster tilting objects

Throughout this section we fix $w \in C$.

It is known that $\operatorname{id}_{\Lambda_w} \Lambda_w \leq 1$, i.e., Λ_w is Gorenstein of dimension at most 1. Consequently, if $\operatorname{Sub} \Lambda_w$ is the full subcategory of the category of Λ_w-modules formed by the submodules of projective Λ_w-modules, then $\operatorname{Sub} \Lambda_w$ is a Frobenius category, i.e., $\operatorname{Sub} \Lambda_w$ has enough projectives and injectives and in $\operatorname{Sub} \Lambda_w$ the projectives and the injectives coincide. We may form its stable category $\underline{\operatorname{Sub}} \Lambda_w$, which is a Hom-finite triangulated category. Moreover, $\underline{\operatorname{Sub}} \Lambda_w$ is 2-Calabi–Yau, i.e., for all $X, Y \in \underline{\operatorname{Sub}} \Lambda_w$ we have isomorphisms

$$\mathrm{D}\operatorname{Ext}^1_{\underline{\operatorname{Sub}} \Lambda_w}(X, Y) \simeq \operatorname{Ext}^1_{\underline{\operatorname{Sub}} \Lambda_w}(Y, X),$$

which are natural both in X and Y, where $\mathrm{D} := \operatorname{Hom}_k(-, k)$. One may show that if Q is not of type $\mathbb{A}_n$ and $w = c^2$, where c is the Coxeter element in C, then $\underline{\operatorname{Sub}} \Lambda_w$ is equivalent to the cluster category associated with Q.

Now we fix a reduced sequence $\omega = (i_1, \ldots, i_l) \in Q_0^l$, $l \in \mathbb{N}$, such that $w = w(\omega)$. We define a Λ_w-module M_ω by

$$M_\omega := \bigoplus_{j \in [1,l]} M_\omega^j,$$

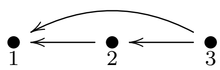
where

$$M_\omega^j := P_{i_j}/(I_{i_1} \cdots I_{i_j} P_{i_j}) \qquad (j \in [1, l])$$

and

$$P_i := \Lambda e_i \qquad (i \in Q_0).$$

Then M_ω is a cluster tilting object in $\underline{\operatorname{Sub}} \Lambda_w$, i.e. $\operatorname{Ext}^1_{\underline{\operatorname{Sub}} \Lambda_w}(M_\omega, M_\omega) = 0$ and if $\operatorname{Ext}^1_{\underline{\operatorname{Sub}} \Lambda_w}(M_\omega, X) = 0$ for some $X \in \underline{\operatorname{Sub}} \Lambda_w$, then $X \in \operatorname{add} M_\omega$. For example, if Q is the quiver



,

$w = s_1 s_2 s_3 s_1 s_2$ and $\omega = (1, 2, 3, 1, 2)$, then M_ω^1 and M_ω^2 can be visualized by the diagrams

$$1 \qquad \text{and} \qquad \begin{matrix} 2 \\ 1 \end{matrix} \quad ,$$

while $M_\omega^3 = \Lambda_w e_1$, $M_\omega^4 = \Lambda_w e_2$ and $M_\omega^5 = \Lambda_w e_3$.

Now we describe $\operatorname{End}_{\underline{\operatorname{Sub}} \Lambda_w}(M_\omega)$. We need a function $\psi : [1, l] \to [1, l+1]$ defined by

$$\psi(j) := \min\{p \in [j+1, l] : i_p = i_j\} \qquad (j \in [1, l]),$$

where $\min \varnothing := l + 1$, i.e. $\psi(j)$ is the number of the next occurrence of i_j in ω (or $\psi(j) := l+1$ if there is no more i_j in ω). Now we define a quiver Δ'. First, we put $\Delta'_0 = [1, l]$. Next, for each $j \in [1, l]$ and

$a \in Q_1$ such that $sa = i_j$ and $\{p \in [j+1, \psi(j)-1] : i_p = ta\} \neq \varnothing$ we have an arrow

$$a_j : j \to \max\{p \in [j+1, \psi(j)-1] : i_p = ta\}$$

in Δ' (here for an arrow a we denote by sa and ta its starting and terminating vertices, respectively). Similarly, for each $j \in [1, l]$ and each $a \in Q_1$ such that $ta = i_j$ and $\{p \in [j+1, \psi(j)-1] : i_p = sa\} \neq \varnothing$ we have an arrow

$$a_j^* : j \to \max\{p \in [j+1, \psi(j)-1] : i_p = sa\}$$

in Δ'. Finally, for each $j \in [1, l]$ such that $\psi(j) \neq l+1$ we have an arrow $\alpha_j : \psi(j) \to j$ in Δ'. We put

$$\Delta := \Delta' \setminus \{j \in [1, l] : \psi(j) = l+1\}.$$

Now let $\mathcal{A}$ be the set of the pairs (j, a) such that $j \in [1, l]$, $a \in Q_1$ and $a_j, a^*_{ta_j} \in \Delta_1$ (in particular, this means that they are defined). Then there exists $m_{j,a} \in \mathbb{N}_+$ such that $ta^*_{ta_j} = \psi^{m_{j,a}}(j)$, and we put

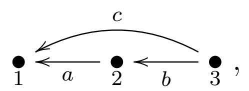
$$c(j, a) := \alpha_j \cdots \alpha_{\psi^{m_{j,a}-1}(j)} a^*_{ta_j} a_j.$$

We define $\mathcal{A}^*$ and $c^*(j, a)$ for all $(j, a) \in \mathcal{A}^*$, dually. Finally we put

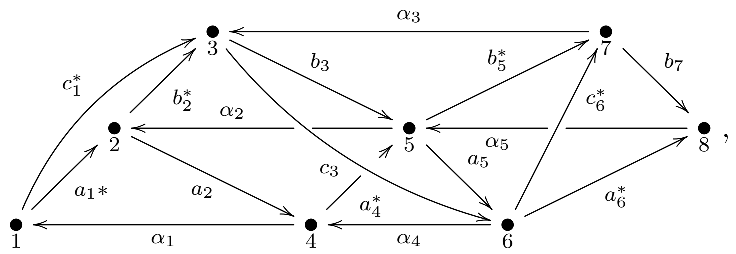
$$W = \sum_{(j,a) \in \mathcal{A}} c(j, a) - \sum_{(j,a) \in \mathcal{A}^*} c^*(j, a).$$

Then $\operatorname{End}_{\underline{\mathrm{Sub}}\,\Lambda_w}(M_\omega)$ is isomorphic to the Jacobian algebra associated with (Δ, W).

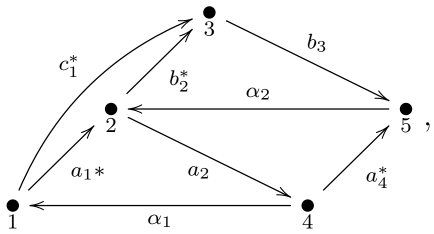
For example, if Q is the quiver



$w = s_1 s_2 s_3 s_1 s_2 s_1 s_3 s_2$ and $\omega = (1, 2, 3, 1, 2, 1, 3, 2)$, then Δ' is the following quiver



Δ is the following quiver



and

$$W = \alpha_2 a_4^* a_2 - \alpha_1 a_2 a_1^* - \alpha_2 b_3 b_2^*.$$

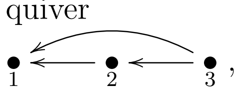
Consequently, $\operatorname{End}_{\underline{\operatorname{Sub}}\,\Lambda_w}(M_\omega)$ is isomorphic to the path algebra of Δ bound by the relations

$$a_2 a_1^*,\ \alpha_1 a_2,\ \alpha_2 a_4^* - a_1^* \alpha_1,\ a_2 \alpha_2,\ a_4^* a_2 - b_3 b_2^*,\ b_2^* \alpha_2,\ \alpha_2 b_3.$$

4. Layers associated with elements in Coxeter groups

Similarly as in the previous section we fix $w \in C$ and a reduced sequence $\omega = (i_1, \dots, i_l) \in Q_0^l$, $l \in \mathbb{N}$, such that $w = w(\omega)$. Let $\psi : [1, l] \to [1, l+1]$ be the function defined in the previous section. If $j \in [1, l]$ and there is no $i \in [1, l]$ such that $j = \psi(i)$, then we put $L_\omega^j := M_\omega^j$. Otherwise, there is unique $i \in [1, l]$ such that $j = \psi(i)$ and we put $L_\omega^j := \operatorname{Ker} f$, where $f : M_\omega^j \to M_\omega^i$ is a homomorphism, which induces an isomorphism of the tops. We call the above modules the layers of M_ω.

For example, if Q is the quiver



$w = s_1 s_2 s_3 s_1 s_3$ and $\omega = (1, 2, 3, 1, 3)$, then L_ω^1, L_ω^2, L_ω^3, L_ω^4 and L_ω^5 can be visualized by the diagrams

$$1, \quad \begin{array}{c} 2 \\ 1 \end{array}, \quad \begin{array}{cccc} & 3 & & \\ 1 & & 2 & \\ & & & 1 \end{array},$$

$$\begin{array}{ccccc} 2 & & 3 & & \\ & 1 & & 2 & \\ & & & & 1 \end{array} \quad \text{and} \quad 2,$$

respectively. Similarly, if $w = s_1 s_2 s_3 s_2 s_1 s_3$ and $\omega = (1, 2, 3, 2, 1, 3)$, then L_ω^1, L_ω^2, L_ω^3, L_ω^4, L_ω^5 and L_ω^6 can be visualized by the diagrams

$$1, \quad \begin{array}{c} 2 \\ 1 \end{array}, \quad \begin{array}{cccc} & 3 & & \\ 1 & & 2 & \\ & & & 1 \end{array}, \quad \begin{array}{c} 3 \\ 1 \end{array}$$

$$\begin{array}{ccccccc} & & 2 & & 3 & & \\ & 3 & & 1 & & 2 & \\ 1 & & & & & & 1 \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & 2 & & 3 \\ & 3 & & 1 & \\ 1 & & & & \end{array},$$

respectively. Observe that in the former example the layers are kQ-modules, while in the latter one L_ω^5 and L_ω^6 are not.

Theorem. *Let w and ω be as above. Then*

$$\operatorname{End}_{\Lambda_w}(L_\omega^j) \simeq k \qquad \text{and} \qquad \operatorname{Ext}_{\Lambda_w}(L_\omega^j, L_\omega^j) = 0$$

for all $j \in [1, l]$. Moreover, for each $j \in [1, l]$ there exists a unique indecomposable kQ-module L'^j_ω such that $\mathbf{dim}\, L_\omega^j = \mathbf{dim}\, L'^j_\omega$.

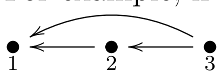
For example, in the latter example L'^5_ω is given by the diagram

$$\begin{array}{ccccccc} & 3 & & & & & \\ 1 & & 2 & & 3 & & \\ & & & 1 & & 2 & \\ & & & & & & 1 \end{array} .$$

Observe that $\operatorname{Ext}^1_{kQ}(L'^5_\omega, L'^5_\omega) \neq 0$.

5. Connection with tilting theory

Throughout this section we fix an admissible ordering $\sigma : [1, n] \to Q_0$ and put $\gamma = (\sigma(1), \ldots, \sigma(n))$. Note that $w(\gamma)$ is the Coxeter element. We say that a reduced sequence $\omega \in Q_0^l$, $l \in \mathbb{N}$, is sortable if $\omega = (\gamma^{(0)}, \ldots, \gamma^{(r)})$, where $\gamma^{(0)}$ is a subsequence of γ and $\gamma^{(i)}$ is a subsequence of $\gamma^{(i-1)}$ for each $i \in [1, r]$. For example, if Q is the quiver



and σ is the identity map, then $(1, 2, 3, 1, 3)$ is sortable (with $\gamma^{(0)} = (1, 2, 3)$ and $\gamma^{(1)} = (1, 3)$), while $(1, 2, 3, 2, 1, 3)$ is not. In general, if ω is sortable, then L_ω^j is a kQ-module for each $j \in [1, l]$. Observe that if ω and ω' are sortable and $w(\omega) = w(\omega')$, then $\omega = \omega'$. Consequently, we may speak about sortable elements in C instead of sortable sequences. Reading showed that there if Q is a Dynkin quiver, there there is a bijection between sortable elements in C and the clusters.

Now we fix a sortable sequence $\omega = (i_1, \ldots, i_l) \in Q_0^l$, $l \in \mathbb{N}$, such that $l \geq n$ and $i_j = \gamma(j)$ for each $j \in [1, n]$ (we call such sortable sequences admissible). This means that $L_\omega^j = (kQ)e_j$ for each $j \in [1, n]$. We define the subsets $I_n, \ldots, I_t \subseteq [1, l]$ together with bijections $\sigma_j : [1, n] \to I_j$, $j \in [n, t]$, by the following rules:

(1) $I_n := [1, n]$ and σ_n is the identity map,
(2) if $j > n$, then

$$I_j := I_{j-1} \setminus \{\sigma_{j-1}(i_j)\} \cup \{j\}$$

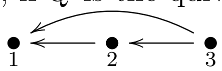
and

$$\sigma_j(i) := \begin{cases} \sigma_{j-1}(i) & i \neq i_j \\ j & i = i_j \end{cases} \qquad (i \in [1, n]).$$

Finally, we put

$$T^j_\omega := \bigoplus_{i\in[1,n]} L^{\sigma_j(i)}_\omega \qquad (j \in [n,t])$$

and $T_\omega = T^t_\omega$. For example, if Q is the quiver



σ is the identity map and $\omega = (1, 2, 3, 1, 3)$, then

$$T^3_\omega = L^1_\omega \oplus L^2_\omega \oplus L^3_\omega, \qquad T^4_\omega = L^4_\omega \oplus L^2_\omega \oplus L^3_\omega$$

and

$$T^5_\omega = L^4_\omega \oplus L^2_\omega \oplus L^5_\omega.$$

Theorem. *Let $\omega = (i_1, \dots, i_l) \in Q^l_0$, $l \in \mathbb{N}$, be an admissible sortable sequence. Then we have the following:*

(1) *For each $j \in [n+1, t]$ there exists an exact sequence of the form*

$$0 \to L^{\sigma_{j-1}(i_j)}_\omega \xrightarrow{f_j} L'_j \to L^j_\omega \to 0,$$

such that f is a minimal left $\operatorname{add}(\bigoplus_{i\in I_j\setminus\{j\}} L^i_\omega)$*-approximation.*

(2) *T_ω is a tilting kQ-module and L^1_ω, ..., L^t_ω are representatives of the indecomposable modules in* $\operatorname{Sub} T_\omega$.

Recall that $\operatorname{Sub} T$ is a torsion free class for a tilting module T. Thus the following can be seen as a converse of the second part of the above theorem.

Theorem. *Let $\mathcal{F}$ be a torsion free class in* $\operatorname{mod} kQ$ *of finite representation type containing kQ. Then there exists a unique admissible sortable sequence ω such that $\mathcal{F} = \operatorname{Sub} T_\omega$.*

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