HOCHSCHILD COHOMOLOGY AND HOMOLOGY OF ALGEBRAS

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Throughout the talk k is a field and Λ is an indecomposable k-algebra.

1. Classical definitions

We first present definition of Hochschild cohomology groups from his classical paper [12].

Let B be a bimodule over Λ. We denote by $\mathbb{H}^B$ the sequence

$$0 \xrightarrow{\partial^{-1}} \operatorname{Hom}_k(\Lambda^{\otimes 0}, B) \xrightarrow{\partial^0} \operatorname{Hom}_k(\Lambda^{\otimes 1}, B) \xrightarrow{\partial^1} \operatorname{Hom}_k(\Lambda^{\otimes 2}, B) \to \cdots ,$$

where $\Lambda^{\otimes n} := \underbrace{\Lambda \otimes_k \cdots \otimes_k \Lambda}_{n \text{ times}}$ for $n \in \mathbb{N}$ (in particular, $\Lambda^0 := k$) and

$$\begin{aligned}(\partial^n f)(\lambda_1 \otimes \cdots \otimes \lambda_{n+1}) &:= \lambda_1 \cdot f(\lambda_2 \otimes \cdots \otimes \lambda_{n+1}) \\ &+ \sum_{i \in [1,n]} (-1)^i \cdot f(\lambda_1 \otimes \cdots \otimes \lambda_{i-1} \otimes \lambda_i \cdot \lambda_{i+1} \otimes \lambda_{i+2} \otimes \cdots \otimes \lambda_{n+1}) \\ &+ (-1)^{n+1} f(\lambda_1 \otimes \cdots \otimes \lambda_n) \cdot \lambda_{n+1}\end{aligned}$$

for $n \in \mathbb{N}$, $f \in \operatorname{Hom}_k(\Lambda^{\otimes n}, B)$ and $\lambda_1, \ldots, \lambda_{n+1} \in \Lambda$. Note that we have the canonical isomorphism $\operatorname{Hom}_k(\Lambda^{\otimes 0}, B) \simeq B$ sending $f \in \operatorname{Hom}_k(\Lambda^{\otimes 0}, B)$ to $f(1)$, and under this isomorphism ∂^0 is given by

$$(\partial^0 b)(\lambda) = \lambda \cdot b - b \cdot \lambda$$

for each $b \in B$ and $\lambda \in \Lambda$. One checks that $\mathbb{H}^B$ is a complex, i.e., $\partial^n \circ \partial^{n-1} = 0$ for each $n \in \mathbb{N}$ (it will also follow from our considerations in Section 2) and for $n \in \mathbb{N}$ we define the n-th Hochschild cohomology group of Λ with coefficients in B by the formula

$$\operatorname{HH}^n(\Lambda, B) := \operatorname{Ker} \partial^n / \operatorname{Im} \partial^{n-1}.$$

We have the following homological version of the above definition (we note, however, that it was not defined by Hochschild). Let $\mathbb{H}_B$ be the sequence

$$\cdots \to B \otimes_k \Lambda^{\otimes 2} \xrightarrow{\partial_2} B \otimes_k \Lambda^{\otimes 1} \xrightarrow{\partial_1} B \otimes_k \Lambda^{\otimes 0} \xrightarrow{\partial_0} 0,$$

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where

$$\partial_n(b\otimes\lambda_1\otimes\cdots\otimes\lambda_n) := b\cdot\lambda_1\otimes\lambda_2\otimes\cdots\otimes\lambda_n + \sum_{i\in[1,n-1]}(-1)^i\cdot b\otimes\lambda_1\otimes\lambda_{i-1}\otimes\lambda_i\cdot\lambda_{i+1}\otimes\lambda_{i+2}\otimes\cdots\otimes\lambda_n + (-1)^n\cdot\lambda_n\cdot b\otimes\lambda_1\otimes\cdots\otimes\lambda_{n-1}$$

for $n\in\mathbb{N}$, $b\in B$ and $\lambda_1,\ldots,\lambda_n\in\Lambda$. Again $\mathbb{H}_B$ is a complex, i.e., $\partial_n\circ\partial_{n+1}=0$ for each $n\in\mathbb{N}$, and for $n\in\mathbb{N}$ we define the n-th Hochschild homology group of Λ with coefficients in B by the formula

$$\mathrm{HH}_n(\Lambda,B) := \operatorname{Ker}\partial_n/\operatorname{Im}\partial_{n+1}.$$

Observe that

$$\mathrm{HH}^0(\Lambda,B) = \{b\in B : \lambda\cdot b = b\cdot\lambda \text{ for each } \lambda\in\Lambda\}.$$

Consequently, $\mathrm{HH}^0(\Lambda,\Lambda)$ is just the center $Z(\Lambda)$ of Λ. In particular, $\mathrm{HH}^0(\Lambda,\Lambda)=\Lambda$ if and only if Λ is commutative. On the other hand,

$$\mathrm{HH}_0(\Lambda,B) = B/\{\lambda\cdot b - b\cdot\lambda : \lambda\in\Lambda,\, b\in B\},$$

hence again $\mathrm{HH}_0(\Lambda,\Lambda)=\Lambda$ if and only if Λ is commutative.

By a k-derivation of Λ on B we mean every k-linear map $d:\Lambda\to B$ such

$$d(\lambda_1\cdot\lambda_2) = d(\lambda_1)\cdot\lambda_2 + \lambda_1\cdot d(\lambda_2)$$

for all $\lambda_1,\lambda_2\in\Lambda$. A derivation d is called inner, if there exists $b\in B$ such that

$$d(\lambda) = \lambda\cdot b - b\cdot\lambda$$

for each $\lambda\in\Lambda$. One checks that every map of the above form is a derivation. We denote by $\mathrm{Der}_k(\Lambda,B)$ and $\mathrm{Der}_k^*(\Lambda,B)$ the space of the k-derivations and the inner k-derivations, respectively. Then

$$\operatorname{Ker}\partial^1 = \mathrm{Der}_k(\Lambda,B) \qquad\text{and}\qquad \operatorname{Im}\partial^0 = \mathrm{Der}_k^*(\Lambda,B),$$

thus

$$\mathrm{HH}^1(\Lambda,B) = \mathrm{Der}_k(\Lambda,B)/\mathrm{Der}_k^*(\Lambda,B).$$

As an example we calculate $\mathrm{HH}^1(k[X],k[X])$. We immediately get that $\mathrm{Der}_k^*(k[X],k[X])=0$, since $k[X]$ is commutative. On the other, for each $f\in k[X]$ we define a derivation $d_f : k[X]\to k[X]$ by the formula $d_f(g) := g'\cdot f$ for $g\in k[X]$. If we define $\Psi : k[X]\to\mathrm{Der}_k(k[X],k[X])$ by $\Psi(f) := d_f$ for $f\in k[X]$, then one easily checks that Ψ is an isomorphism. In other words, $\mathrm{HH}^1(k[X],k[X]) = k[X]$.

Now consider the quiver

$$Q:\ \underset{2}{\bullet} \xleftarrow{\ \alpha\ } \underset{1}{\bullet} \xrightarrow{\ \beta\ } \underset{3}{\bullet}$$

and fix $d\in\mathrm{Der}_k(kQ,kQ)$. Using the equality $d(e_2)=e_2\cdot d(e_2)+d(e_2)\cdot e_2$ we get that there exists $a'\in k$ such that $d(e_2)=a'\cdot\alpha$. Analogously, we get that $d(e_3)=a''\cdot\beta$ for some $a''\in k$. Moreover, the equality

$1 = e_1 + e_2 + e_3$ implies that $d(e_1) = -(a' \cdot \alpha + a'' \cdot \beta)$. Next, using the equalities

$$d(e_2) \cdot \alpha + e_2 \cdot d(\alpha) = d(\alpha) = d(\alpha) \cdot e_1 + \alpha \cdot d(e_1),$$

we get that there exists $b' \in k$ such that $d(\alpha) = b' \cdot \alpha$. Similarly, $d(\beta) = b'' \cdot \beta$ for some $b'' \in k$. Thus, if

$$x := a' \cdot \alpha + a'' \cdot \beta - b' \cdot e_2 - b'' \cdot e_3,$$

then $d(y) = y \cdot x - x \cdot y$ for each $y \in kQ$, hence $\mathrm{HH}^1(kQ, kQ) = 0$. This result is not surprising, since we have the following.

Theorem (Happel [11])**.** *Let k be algebraically closed and Q a finite quiver without oriented cycles. Then $\mathrm{HH}^1(kQ, kQ) = 0$ if and only if Q is a tree.*

We remark, that if Q has an oriented cycle, then one can easily construct a derivation of kQ on kQ, which is not inner.

2. Modern approach

Consider the sequence

$$\mathbb{S} : \cdots \to \Lambda^{\otimes 4} \xrightarrow{d_2} \Lambda^{\otimes 3} \xrightarrow{d_1} \Lambda^{\otimes 2} \xrightarrow{d_0} \Lambda^{\otimes 1} \xrightarrow{d_{-1}} 0,$$

where

$$\begin{aligned} & d_n(\lambda_0 \otimes \cdots \otimes \lambda_{n+1}) \\ & \qquad := \sum_{i \in [0,n]} (-1)^i \cdot \lambda_0 \otimes \cdots \otimes \lambda_{i-1} \otimes \lambda_i \cdot \lambda_{i+1} \otimes \lambda_{i+2} \otimes \cdots \otimes \lambda_{n+1} \end{aligned}$$

for $n \in \mathbb{N}$ and $\lambda_0, \ldots, \lambda_{n+1} \in \Lambda$. It is a sequence of Λ-Λ-bimodules, if for $n \in \mathbb{N}_+$ and $\lambda, \lambda', \lambda_1, \ldots, \lambda_n \in \Lambda$ we put

$$\lambda \cdot (\lambda_1 \otimes \cdots \otimes \lambda_n) \cdot \lambda' := \lambda \cdot \lambda_1 \otimes \lambda_2 \cdots \otimes \lambda_{n-1} \otimes \lambda_n \cdot \lambda'.$$

For $n \in \mathbb{N}$ we define $s_n : \Lambda^{\otimes(n+1)} \to \Lambda^{\otimes(n+2)}$ by the formula $s_n(x) := 1 \otimes x$ for $x \in \Lambda^{\otimes(n+1)}$. Moreover, we denote by s_{-1} the zero map $\Lambda \to 0$. One verifies directly that

$$d_n \circ s_n + s_{n-1} \circ d_{n-1} = \mathrm{Id}$$

for each $n \in \mathbb{N}$. As a first consequence we obtain the following.

Lemma 2.1. *$\mathbb{S}$ is a complex.*

Proof. It is obvious that $d_{-1} \circ d_0 = 0$. If $n \in \mathbb{N}$, then

$$d_n \circ d_{n+1} \circ s_{n+1} = d_n - d_n \circ s_n \circ d_n = d_n - d_n + s_{n-1} \circ d_{n-1} \circ d_n.$$

By induction, $d_{n-1} \circ d_n = 0$, hence $d_n \circ d_{n+1} \circ s_{n+1} = 0$. Since both d_n and d_{n+1} are homomorphisms of left Λ-modules, we get

$$(d_n \circ d_{n+1})(\lambda \otimes x) = \lambda \cdot (d_n \circ d_{n+1})(1 \otimes x) = \lambda \cdot (d_n \circ d_{n+1} \circ s_{n+1})(x) = 0$$

for all $\lambda \in \Lambda$ and $x \in \Lambda^{\otimes(n+1)}$, hence the claim follows. $\square$

Lemma 2.2. $\mathbb{S}$ *is exact.*

Proof. We already know that $\operatorname{Im} d_n \subseteq \operatorname{Ker} d_{n-1}$ for each $n \in \mathbb{N}$. On the other hand, if $n \in \mathbb{N}$ and $x \in \operatorname{Ker} d_{n-1}$, then $x = (d_n \circ s_n)(x) \in \operatorname{Im} d_n$. □

Proposition 2.3. $\mathbb{S}$ *is a projective resolution of* Λ *as a* Λ-Λ-*bimodule.*

Proof. It is enough to observe that $\Lambda^{\otimes(n+2)}$ is a projective Λ-Λ-bimodule for each $n \in \mathbb{N}$. □

Theorem 2.4. *If* B *is a* Λ-Λ-*bimodule, then*

$$\mathrm{HH}^n(\Lambda, B) \simeq \mathrm{Ext}^n_{\Lambda-\Lambda}(\Lambda, B) \qquad \textit{and} \qquad \mathrm{HH}_n(\Lambda, B) \simeq \mathrm{Tor}_n^{\Lambda-\Lambda}(\Lambda, B)$$

for each $n \in \mathbb{N}$.

Proof. Let $\mathbb{S}'$ be the sequence

$$\mathbb{S}' : \cdots \to \Lambda^{\otimes 4} \xrightarrow{d_2} \Lambda^{\otimes 3} \xrightarrow{d_1} \Lambda^{\otimes 2} \to 0.$$

One easily checks that $\mathrm{Hom}_{\Lambda-\Lambda}(\mathbb{S}', B)$ is isomorphic to $\mathbb{H}^B$ and $B \otimes_{\Lambda-\Lambda} \mathbb{S}'$ is isomorphic to $\mathbb{H}_B$, hence the claim follows. □

Now we apply the above theorem in order to calculate the Hochschild (co)homology groups for $\Lambda := k[X]/(X^a)$, $a \in \mathbb{N}_+$. Let $\mathbb{P}$ be the following sequence

$$\cdots \to \Lambda \otimes_k \Lambda \xrightarrow{\cdot v} \Lambda \otimes_k \Lambda \xrightarrow{\cdot w} \Lambda \otimes_k \Lambda \xrightarrow{\cdot v} \Lambda \otimes_k \Lambda \xrightarrow{\mu} \Lambda \to 0,$$

where $v := X \otimes 1 - 1 \otimes X$, $w := \sum_{i \in [0,a-1]} X^i \otimes X^{a-1-i}$ and $\mu(\lambda_1 \otimes \lambda_2) := \lambda_1 \cdot \lambda_2$ for $\lambda_1, \lambda_2 \in \Lambda$. One checks that $\mathbb{P}$ is an exact sequence, hence $\mathbb{P}$ is a projective resolution of Λ as a Λ-Λ-bimodule. If $\mathbb{P}'$ is the sequence

$$\cdots \to \Lambda \otimes_k \Lambda \xrightarrow{\cdot v} \Lambda \otimes_k \Lambda \xrightarrow{\cdot w} \Lambda \otimes_k \Lambda \xrightarrow{\cdot v} \Lambda \otimes_k \Lambda \to 0,$$

then $\Lambda \otimes_{\Lambda-\Lambda} \mathbb{P}'$ and $\mathrm{Hom}_{\Lambda-\Lambda}(\mathbb{P}', \Lambda)$ equal

$$\cdots \to \Lambda \xrightarrow{0} \Lambda \xrightarrow{\cdot a \cdot X^{a-1}} \Lambda \xrightarrow{0} \Lambda \to 0$$

and

$$0 \to \Lambda \xrightarrow{0} \Lambda \xrightarrow{\cdot a \cdot X^{a-1}} \Lambda \xrightarrow{0} \Lambda \to \cdots,$$

respectively. Consequently,

$$\mathrm{HH}_n(\Lambda, \Lambda) = \begin{cases} \Lambda & n = 0, \\ \Lambda/(a \cdot X^{a-1}) & n \in 2 \cdot \mathbb{N}_+ - 1, \\ \mathrm{Ann}_\Lambda(a \cdot X^{a-1}) & n \in 2 \cdot \mathbb{N}_+, \end{cases}$$

and

$$\mathrm{HH}^n(\Lambda, \Lambda) = \begin{cases} \Lambda & n = 0, \\ \mathrm{Ann}_\Lambda(a \cdot X^{a-1}) & n \in 2 \cdot \mathbb{N}_+ - 1, \\ \Lambda/(a \cdot X^{a-1}) & n \in 2 \cdot \mathbb{N}_+, \end{cases}$$

for each $n \in \mathbb{N}$. In particular, $\mathrm{HH}^n(\Lambda, \Lambda) \neq 0 \neq \mathrm{HH}_n(\Lambda, \Lambda)$ for each $n \in \mathbb{N}$ provided $a \geq 2$. More generally, Holm proved [13] that

$$\mathrm{HH}_n(\Lambda, \Lambda) = \begin{cases} \Lambda & n = 0, \\ \Lambda/(f') & n \in 2 \cdot \mathbb{N}_+ - 1, \\ \mathrm{Ann}_\Lambda(f') & n \in 2 \cdot \mathbb{N}_+, \end{cases}$$

and

$$\mathrm{HH}^n(\Lambda, \Lambda) = \begin{cases} \Lambda & n = 0, \\ \mathrm{Ann}_\Lambda(f') & n \in 2 \cdot \mathbb{N}_+ - 1, \\ \Lambda/(f') & n \in 2 \cdot \mathbb{N}_+, \end{cases}$$

for each $n \in \mathbb{N}$, provided $\Lambda = k[X]/(f)$ for $f \in k[X]$.

3. Vanishing

Lemma 3.1. *Let*

$$\eta : 0 \to P_{n+2} \xrightarrow{d_{n+1}} P_{n+1} \xrightarrow{d_n} P_n \to \cdots \xrightarrow{d_0} P_0 \to 0$$

be an exact sequence of Λ-Λ*-bimodules, such that* $P_0, \ldots, P_n$ *are projective as right* Λ*-modules. If* M *is a left* Λ*-module, then the sequence* $\eta \otimes_\Lambda M$ *is exact.*

Proof. We prove by induction on l the following two claims:

(1) $\mathrm{Im}\, d_l$ is a projective right Λ-module for each $l \in [0, n]$,
(2) the sequence $0 \to \mathrm{Im}\, d_l \to P_l \to \mathrm{Im}\, d_{l-1} \to 0$ splits as a sequence of right Λ-modules for each $l \in [1, n+1]$.

The second claim will imply that the sequence

$$0 \to \mathrm{Im}\, d_l \otimes_\Lambda M \to P_l \otimes_\Lambda M \to \mathrm{Im}\, d_{l-1} \otimes_\Lambda M \to 0,$$

is exact for each $l \in [1, n+1]$, which immediately implies that $\eta \otimes_\Lambda M$ is exact.

Now $\mathrm{Im}\, d_0 = P_0$ is a projective right Λ-module by assumption, thus assume $l > 0$. By induction $\mathrm{Im}\, d_{l-1}$ is a projective Λ-module, hence the sequence $0 \to \mathrm{Im}\, d_l \to P_l \to \mathrm{Im}\, d_{l-1} \to 0$ splits as a sequence of right Λ-modules. Moreover, if $l < n+1$, then P_l is a projective right Λ-module by assumption, hence $\mathrm{Im}\, d_l$ is a projective right Λ-module as well. $\square$

Theorem 3.2. *If* $\mathbb{P}$ *is a projective resolution of* Λ *as a* Λ-Λ*-bimodule, then* $\mathbb{P} \otimes_\Lambda M$ *is a projective resolution of* M *for each left* Λ*-module* M*.*

Proof. Note that $\mathbb{P} \otimes_\Lambda M$ is a sequence of projective left Λ-modules. Moreover, the above lemma implies that $\mathbb{P} \otimes_\Lambda M$ is an exact sequence. $\square$

Corollary 3.3. $\operatorname{lgldim}\Lambda \leq \operatorname{pdim}_{\Lambda-\Lambda}\Lambda$ *and* $\operatorname{rgldim}\Lambda \leq \operatorname{pdim}_{\Lambda-\Lambda}\Lambda$. *In particular, if* Λ *is left and right Noetherian* (*for example,* $\dim_k \Lambda < \infty$), *then*

$$\operatorname{gldim}\Lambda \leq \operatorname{pdim}_{\Lambda-\Lambda}\Lambda.$$

Theorem 3.4. *If* k *is algebraically closed and* $\dim_k \Lambda < \infty$, *then* $\operatorname{gldim}\Lambda = \operatorname{pdim}_{\Lambda-\Lambda}\Lambda$.

Proof. Let

$$\mathbb{P} : \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \Lambda \to 0$$

be a minimal projective resolution of Λ as a Λ-Λ-bimodule (existence of such resolution follows since $\dim_k \Lambda < \infty$). The minimality assumption implies that

$$\operatorname{Im} d_n \subseteq \operatorname{rad}\Lambda \cdot P_{n-1} + P_{n-1} \cdot \operatorname{rad}\Lambda$$

for each $n \in \mathbb{N}_+$ (here we use the assumption that k is algebraically closed). If S is a simple left Λ-module, then

$$\operatorname{Im}(d_n \otimes_\Lambda S) \subseteq \operatorname{rad}\Lambda \cdot P_{n-1} \otimes_\Lambda S + P_{n-1} \cdot \operatorname{rad}\Lambda \otimes_\Lambda S \subseteq \operatorname{rad}\Lambda \cdot (P_{n-1} \otimes_\Lambda S)$$

for each $n \in \mathbb{N}_+$ (we use that $\operatorname{rad}\Lambda \cdot S = 0$), hence $\mathbb{P} \otimes_\Lambda S$ is a minimal projective resolution of S. Moreover, if P is a projective Λ-Λ-bimodule and $P \otimes_\Lambda S = 0$ for each simple left Λ-module S, then we prove by induction that $P \otimes_\Lambda M = 0$ for each left Λ-module M, hence $P = 0$. Consequently,

$$\begin{aligned} \operatorname{gldim}\Lambda &= \sup\{n \in \mathbb{N} : P_n \otimes S \neq 0 \text{ for each simple left } \Lambda\text{-module } S\} \\ &= \sup\{n \in \mathbb{N} : P_n \neq 0\} = \operatorname{pdim}_{\Lambda-\Lambda}\Lambda, \end{aligned}$$

hence the claim follows. □

Corollary 3.5. *If* k *is algebraically closed and* $\dim_k \Lambda < \infty$, *then* $\mathrm{HH}^n(\Lambda, B) = 0 = \mathrm{HH}_n(\Lambda, B)$ *for each* Λ-Λ-*bimodule* B *and* $n > \operatorname{gldim}\Lambda$.

Inspired by the above result Happel [11] asked a question, if the condition $\mathrm{HH}^n(\Lambda, \Lambda) = 0$ for $n \gg 0$ implies that $\operatorname{gldim}\Lambda < \infty$. Avramov and Iyengar [1] showed that this is the case if Λ is commutative. On the other hand, Buchweitz, Green, Madsen and Solberg [8] gave a counterexample for a general version. Namely, if $q \in k$ is not a root of unity and

$$\Lambda := k\langle X, Y\rangle/(X^2, X \cdot Y - q \cdot Y \cdot X, Y^2),$$

then $\mathrm{HH}^n(\Lambda, \Lambda) = 0$ for each $n \geq 3$, while $\operatorname{gldim}\Lambda = \infty$. However, the following conjecture is still open.

Conjecture. *If* $\dim_k \Lambda < \infty$ *and* $\mathrm{HH}_n(\Lambda, \Lambda) = 0$ *for each* $n \gg 0$, *then* $\operatorname{gldim}\Lambda < \infty$.

The above conjecture has been verified if Λ is either commutative [2] or monomial [10]. Moreover, under the assumption $\operatorname{char} k = 0$ Bergh and Madsen [5] proved this conjecture in the case when Λ is either Koszul or graded with $\Lambda_0 = k$. Finally, Bergh, Madsen and Han [4] verified the conjecture provided there exists arrows $\alpha_1, \dots, \alpha_t$ in the Gabriel quiver of Λ such that $s\alpha_i = t\alpha_{i-1}$ and $\alpha_i\alpha_{i-1} = 0$ for each $i \in [1, t]$, where $\alpha_0 := \alpha_t$.

We present the proof of the conjecture for quantum complete intersections due to Bergh and Madsen [6].

First observe that if $f : \Lambda \to \Gamma$ is a homomorphism of k-algebras, then we have the induced map $f^{\otimes n} : \Lambda^{\otimes n} \to \Gamma^{\otimes n}$ for each $n \in \mathbb{N}_+$, which induces the map $\mathrm{HH}_n(f) : \mathrm{HH}_n(\Lambda, \Lambda) \to \mathrm{HH}_n(\Gamma, \Gamma)$ for each $n \in \mathbb{N}$. In other words, we obtain functors HH_n, $n \in \mathbb{N}$, from the category of k-algebras to the category of vector spaces.

Theorem 3.6. *If*

$$\Lambda := k\langle X_1, \dots, X_c\rangle/(X_i^{a_i}, X_i \cdot X_j - q_{i,j} \cdot X_j \cdot X_i)$$

for $c \in \mathbb{N}_+$, $a_1, \dots, a_c \geq 2$, *and* $q_{i,j} \in k^\times$, $i < j$, *then* $\mathrm{HH}_n(\Lambda, \Lambda) \neq 0$ *for each* $n \in \mathbb{N}$.

Proof. For each $i \in [1, c]$ we have algebra homomorphisms

$$\iota_c : k[X]/(X^{a_c}) \to \Lambda \qquad \text{and} \qquad \pi_c : \Lambda \to k[X]/(X^{a_c})$$

such that $\pi_c \circ \iota_c = \mathrm{Id}$. This implies that $\mathrm{HH}_n(k[X]/(X^{a_i}), k[X]/(X^{a_i}))$ is a direct summand of $\mathrm{HH}_n(\Lambda, \Lambda)$ for each $n \in \mathbb{N}$ and $i \in [1, c]$. Since $\mathrm{HH}_n(k[X]/(X^a), k[X]/(X^a)) \neq 0$ for each $n \in \mathbb{N}$ and $a \geq 2$, the claim follows. $\square$

4. The Hochschild cohomology ring

Throughout this section we assume that $\dim_k \Lambda < \infty$.

Since $\mathrm{HH}^n(\Lambda) = \mathrm{Ext}^n_{\Lambda-\Lambda}(\Lambda, \Lambda)$ for each $n \in \mathbb{N}$, we get a graded ring

$$\mathrm{HH}^*(\Lambda, \Lambda) := \bigoplus_{n\in\mathbb{N}} \mathrm{HH}^n(\Lambda, \Lambda)$$

with the multiplication given by the Yoneda product.

Theorem 4.1 (Yoneda [16])**.** *Let* Λ, Σ *and* Γ *be* k*-algebras,* A *and* B Λ-Σ*-bimodules, and* C *and* D Σ-Γ*-bimodules. If* A, B, C *and* D *are flat as* Σ*-modules, then*

$$(\eta \otimes_\Sigma D) \circ (A \otimes_\Sigma \theta) = (-1)^{mn} \cdot (B \otimes_\Sigma \theta) \circ (\eta \otimes_\Sigma C)$$

for all $\eta \in \mathrm{Ext}^m_{\Lambda-\Sigma}(A, B)$ *and* $\theta \in \mathrm{Ext}^n_{\Sigma-\Gamma}(C, D)$.

Corollary 4.2. $\mathrm{HH}^*(\Lambda, \Lambda)$ *is graded commutative.*

Observe that $\eta^2 = -\eta^2$ for each $\eta \in \mathrm{HH}^n(\Lambda, \Lambda)$ such that n is odd. In particular, $\eta^2 = 0$ if $\operatorname{char} k \neq 2$. Moreover, $\mathrm{HH}^{2*}(\Lambda, \Lambda) := \bigoplus_{n \in \mathbb{N}} \mathrm{HH}^{2n}(\Lambda, \Lambda)$ is commutative and $\mathrm{HH}^*(\Lambda, \Lambda)$ is commutative if $\operatorname{char} k = 2$. Finally, if $\mathcal{N}$ is the ideal in $\mathrm{HH}^*(\Lambda, \Lambda)$ generated by the homogeneous nilpotent elements, then $\mathrm{HH}^*(\Lambda, \Lambda)/\mathcal{N}$ is commutative.

Theorem 4.3 (Green/Snashall/Solberg [9])**.** *Assume that k is algebraically closed. If there exists $n \in \mathbb{N}_+$ such that $\Omega^n_{\Lambda-\Lambda}(\Lambda) \simeq \Lambda$, then*

$$\mathrm{HH}^* / \mathcal{N} \simeq k[X]$$

and $|X| = \min\{n \in \mathbb{N}_+ : \Omega^n_{\Lambda-\Lambda}(\Lambda) \simeq \Lambda\}$.

We remark that for the above theorem the assumption that Λ is indecomposable is important. Holm [13] showed that

$$\mathrm{HH}^*(\Lambda, \Lambda) = k[X, Y, Z]/(X^a, a \cdot X^{a-1} \cdot Z, Y \cdot X^{a-1}, Y^2)$$

with $|X| = 0$, $|Y| = 1$ and $|Z| = 2$, if $\Lambda := k[X]/(X^a)$ for $a \geq 2$. In particular, $\mathrm{HH}^* / \mathcal{N} \simeq k[Z]$ in this case.

5. Support varieties

Let $\eta \in \mathrm{HH}^n(\Lambda, \Lambda) = \mathrm{Ext}^n_{\Lambda-\Lambda}(\Lambda, \Lambda)$. Then η can be represented by an exact sequence

$$0 \to \Lambda \to K \to P_{n-2} \to \cdots \to P_0 \to \Lambda \to 0,$$

such that $P_0, \ldots, P_{n-2}$ are projective Λ-Λ-bimodules. Indeed, if $f_\eta \in \mathrm{Hom}_{\Lambda-\Lambda}(\Omega^n_{\Lambda-\Lambda}(\Lambda), \Lambda)$ corresponds to η, then we can take as a representative the pushout of the sequence

$$0 \to \Omega^n_{\Lambda-\Lambda}(\Lambda) \to P_{n-1} \to \cdots \to P_0 \to \Lambda \to 0$$

by f_η, where

$$\mathbb{P} : \cdots \to P_2 \to P_1 \to P_0 \to \Lambda \to 0$$

is a minimal projective resolution of Λ. Using Lemma 3.1 we know that $\eta \otimes_\Lambda M$ is an exact sequence of left Λ-modules, i.e., $\eta \otimes_\Lambda M \in \mathrm{Ext}^n_\Lambda(M, M)$. Since $\mathbb{P} \otimes_\Lambda M$ is a projective resolution of M by Theorem 3.2, we get that $\eta \otimes_\Lambda M$ corresponds to $f_\eta \otimes_\Lambda M$. In this way we define a function $\Phi_M : \mathrm{HH}^*(\Lambda, \Lambda) \to \mathrm{Ext}^*_\Lambda(M, M)$, which is a homomorphism of graded algebras.

Now let M and N be left Λ-module. We define left and right actions of $\mathrm{HH}^*(\Lambda, \Lambda)$ on $\mathrm{Ext}^*_\Lambda(M, N)$ by the formulas: $\eta \cdot \theta := \Phi_N(\eta) \circ \theta$ and $\theta \cdot \eta := \theta \circ \Phi_M(\eta)$ for $\eta \in \mathrm{HH}^n(\Lambda, \Lambda)$ and $\theta \in \mathrm{Ext}^m_\Lambda(M, N)$. Theorem 4.1 implies the following.

Theorem 5.1. *If $\eta \in \mathrm{HH}^n(\Lambda, \Lambda)$ and $\theta \in \mathrm{Ext}^m_\Lambda(M, N)$, then*

$$\eta \cdot \theta = (-1)^{mn} \cdot \theta \cdot \eta.$$

For a graded algebra Γ we denote by $Z_{\mathrm{gr}}(\Gamma)$ the graded center of Γ, i.e., the subring of Γ generated by the homogeneous elements γ such that $\gamma \cdot \gamma' = (-1)^{|\gamma| \cdot |\gamma'|} \cdot \gamma' \cdot \gamma$ for each homogeneous element γ' of Γ.

Corollary 5.2. *For each left Λ-module M the image of Φ_M is contained in $Z_{\mathrm{gr}}(\mathrm{Ext}^*_\Lambda(M,M))$.*

Let $H := \mathrm{HH}^{2*}(\Lambda,\Lambda)$. Then H is commutative. Moreover, if M and N are left Λ-modules, then $\eta\cdot\theta = \theta\cdot\eta$ for each $\eta \in H$ and $\theta \in \mathrm{Ext}^*_\Lambda(M,N)$. We denote by $V_H(M,N)$ the set of the maximal ideals in H which contain $\mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M,N)$.

Lemma 5.3. *If M is a left Λ-module, then*

$$V_H(M,\Lambda/\operatorname{rad}\Lambda) = V_H(M,M) = V_H(\Lambda/\operatorname{rad}\Lambda, M).$$

We put $V_H(M) := V_H(M,M)$ and call it the support variety of M. Since we assume that Λ is indecomposable, $\mathrm{HH}^0(\Lambda) = Z(\Lambda)$ is a local algebra and

$$\mathfrak{m}_{\mathrm{gr}} := \operatorname{rad} Z(\Lambda) \oplus \bigoplus_{n\in\mathbb{N}^+} \mathrm{HH}^{2n}(\Lambda,\Lambda)$$

is the unique graded maximal ideal of H. Consequently, $\mathfrak{m}_{\mathrm{gr}} \in V_H(M)$ for each nonzero left Λ-module M.

Theorem 5.4.

1. *If either $\mathrm{Ext}^n_\Lambda(M,M) = 0$ for all $n \gg 0$ or $\mathrm{pdim}_\Lambda M < \infty$ or $\mathrm{idim}_\Lambda M < \infty$, then $V_H(M) \subseteq \{\mathfrak{m}_{\mathrm{gr}}\}$.*
2. *If M and N are left Λ-modules, then $V_H(M\oplus N) = V_H(M)\cup V_H(N)$.*
3. *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of left Λ-modules, then $V_H(M_i) \subseteq \bigcup_{j\in[1,3]\setminus\{i\}} V_H(M_j)$ for each $i \in [1,3]$.*
4. *If $\mathrm{pdim}_\Lambda M = \infty$, then $V_H(\Omega^n(M)) = V_H(M)$ for each $n \in \mathbb{N}$.*
5. *If Λ is selfinjective, then $V_H(M) = V_H(\tau M)$.*
6. *If Λ is selfinjective and M and N belong to the same component of the stable Auslander–Reiten quiver of Λ, then $V_H(M) = V_H(N)$.*

Let

$$\cdots \to Q_2 \to Q_1 \to Q_0 \to M \to 0$$

be a minimal projective resolution of a left Λ-module M. We define the complexity $\mathrm{cx}\, M$ of M by

$$\mathrm{cx}\, M := \inf\{t \in \mathbb{N} : \text{there exists a real number } a \text{ such that } \dim_k Q_n \le a\cdot n^{t-1} \text{ for each } n \in \mathbb{N}_+\}.$$

One easily checks that $\mathrm{cx}\, M = 0$ if and only if $\mathrm{pdim}_\Lambda M < \infty$. Similarly, $\mathrm{cx}\, M \le 1$ if and only if the sequence $(\dim_k Q_n)$ is bounded. In particular, $\mathrm{cx}\, M = 1$ if M is nonzero and periodic, i.e., there exists $n \in \mathbb{N}_+$ such that $M \simeq \Omega^n_\Lambda(M)$. Moreover, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of left Λ-modules, then $\mathrm{cx}\, M_i \le \max\{\mathrm{cx}\, M_j : j \in [1,3]\setminus\{i\}\}$ for each $i \in [1,3]$. In particular, $\mathrm{cx}\, M \le \mathrm{cx}(\Lambda/\operatorname{rad}\Lambda)$. Obviously, $\mathrm{cx}\, M = \mathrm{cx}\,\Omega^n(M)$ for each $n \in \mathbb{N}$. Finally, we observe that

Theorem 3.2 implies that $\operatorname{cx} M$ is bounded by the complexity of Λ as a Λ-Λ-bimodule.

We say that Λ satisfies the FG condition if H is Noetherian and $\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$ is a finitely generated H-module. By induction we show that if Λ satisfies the FG condition, then $\operatorname{Ext}^*_\Lambda(M, N)$ is a finitely generated H-module for all left Λ-modules M and N.

Theorem 5.5. *Assume that Λ satisfies the FG condition. Then the following hold.*

(1) *Λ is Gorenstein.*
(2) $\operatorname{cx} M = \dim_k V_H(M) < \infty$ *for each nonzero left Λ-module M.*
(3) $V_H(M) \subseteq \{\mathfrak{m}_{\mathrm{gr}}\}$ *if and only if* $\operatorname{pdim}_\Lambda M < \infty$.
(4) $\dim V_H(M) = 1$ *if and only if* $\operatorname{cx} M = 1$ *and if and only if* $\operatorname{pdim} M = \infty$ *and M is eventually periodic.*
(5) *For each homogenous ideal $\mathfrak{a}$ of H there exists a left Λ-module M such that $V_H(M)$ is the set of the maximal ideals $\mathfrak{m}$ of H such that $\mathfrak{a} \subseteq \mathfrak{m}$.*
(6) *If Λ is selfinjective and $V_H(M) = V_1 \cup V_2$ for closed homogeneous sets V_1 and V_2 such that $V_1 \cap V_2 = \{\mathfrak{m}_{\mathrm{gr}}\}$ and $V_1 \neq \{\mathfrak{m}_{\mathrm{gr}}\} \neq V_2$, then there exist left Λ-modules M_1 and M_2 such that $M = M_1 \oplus M_2$.*
(7) *If Λ is selfinjecitve and there exists a left Λ-module M such that $\operatorname{cx} M \geq 3$, then Λ is wild.*
(8) *Λ satisfies the Auslander condition, i.e., for each left Λ-module M there exists $n \in \mathbb{N}$ such that if $\operatorname{Ext}^i_\Lambda(M, N) = 0$ for $i \gg 0$, then $\operatorname{Ext}^i_\Lambda(M, N) = 0$ for each $i \geq n$.*

Bergh and Oppermann [7] proved that if

$$\Lambda := k\langle X_1, \dots, X_c\rangle/(X_i^{a_i}, X_i \cdot X_j - q_{i,j} \cdot X_j \cdot X_i)$$

for $c \in \mathbb{N}_+$, $a_1, \dots, a_c \geq 2$, and $q_{i,j} \in k^\times$, $i < j$, then Λ satisfies the FG condition if and only if $q_{i,j}$ is a root of unity for all i and j. Now let

$$\Lambda := k\langle X, Y\rangle/(X^2, X \cdot Y - q \cdot Y \cdot X, Y^2),$$

where $q \in k^\times$ is not a root of unity. Note that we have an exact sequence

$$\cdots \to \Lambda \xrightarrow{\cdot(X-q^3\cdot Y)} \Lambda \xrightarrow{\cdot(X+q^2\cdot Y)} \Lambda \xrightarrow{\cdot(X-q\cdot Y)} \Lambda \xrightarrow{\cdot(X+Y)} \Lambda/(X+Y) \to 0,$$

which implies that $\Lambda/(X+Y)$ has complexity 1 and is not (eventually) periodic. Moreover, let M_t be the cokernel of the multiplication by $X + (-q)^t \cdot Y$. Then

$$\dim_k \operatorname{Ext}^i_\Lambda(M, M_t) = \begin{cases} 1 & i = 0, t, t+1, \\ 0 & \text{otherwise}, \end{cases}$$

for each $t \in \mathbb{N}_+$, hence Λ does not satisfy the Auslander condition.

6. Representation dimension

By the representation dimension $\operatorname{repdim}\Lambda$ of Λ we mean

$$\operatorname{repdim}\Lambda := \{\operatorname{gldim}\operatorname{End}_\Lambda(M) : M \text{ is a left } \Lambda\text{-module which is both a generator and cogenerator}\}.$$

It is known that $\operatorname{repdim}\Lambda = 0$ is 0 if Λ is semi-simple. Next, $\operatorname{repdim}\Lambda = 2$ if Λ is of finite representation type, but not semi-simple. Finally, $\operatorname{repdim}\Lambda \geq 3$ if Λ is of infinite representation type. Iyama [14] proved that $\operatorname{repdim}\Lambda < \infty$. Moreover, $\operatorname{repdim}\Lambda$ is not greater then the Loewy length of Λ, if Λ is selfinjective. On the other hand, Rouquier [15] proved that

$$\operatorname{repdim}\Lambda \geq \dim(\mathcal{D}^b(\Lambda)/\mathcal{D}^{\mathrm{perf}}(\Lambda)) + 2.$$

Now let Λ be a Gorenstein algebra and denote by $\mathrm{MCM}(\Lambda)$ the category of the maximal Cohen–Macaulay modules, where a left Λ-module M is called maximal Cohen–Macaulay if $\operatorname{Ext}^n_\Lambda(M,\Lambda) = 0$ for each $n \in \mathbb{N}_+$. Then $\mathrm{MCM}(\Lambda)$ is a Frobenius category, hence its stable category $\underline{\mathrm{MCM}}(\Lambda)$ is a triangulated category. Moreover, $\underline{\mathrm{MCM}}(\Lambda)$ is equivalent with $\mathcal{D}^b(\Lambda)/\mathcal{D}^{\mathrm{perf}}(\Lambda)$.

Theorem 6.1 (Bergh/Iyeangar/Krause/Oppermann [3])**.** *If Λ satisfies the FG condition, then*

$$\dim \underline{\mathrm{MCM}}(\Lambda) \geq \operatorname{cx}(\Lambda/\operatorname{rad}\Lambda) - 1.$$

In particular, $\operatorname{repdim}\Lambda \geq \operatorname{cx}(\Lambda/\operatorname{rad}\Lambda) + 1$.

Theorem 6.2. *If k is algebraically closed and Λ satisfies the FG condition, then*

$$\operatorname{repdim}\Lambda \geq \operatorname{Kdim}\mathrm{HH}^*(\Lambda,\Lambda).$$

If

$$\Lambda := k\langle X_1,\dots,X_c\rangle/(X_i^2, X_i\cdot X_j - q_{i,j}\cdot X_j\cdot X_i)$$

for $c \in \mathbb{N}_+$ and roots of unity $q_{i,j} \in k^\times$, $i < j$, then $\operatorname{cx}(\Lambda/\operatorname{rad}\Lambda) = c$. On the other hand, the Loewy length of Λ equals $c+1$, hence

$$\operatorname{repdim}\Lambda = c + 1,$$

which generalizes the original example of Rouquier, who studied the case of the exterior algebras, i.e., $q_{i,j} = -1$ for all $i < j$.

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