DEGREES OF IRREDUCIBLE MORPHISMS

BASED ON THE TALKS BY CLAUDIA CHAIO

Throughout the talk A is a fixed algebra over an algebraically closed field.

Following Liu by the left degree $d_l(f)$ of a homomorphism $f : X \to Y$ we mean the minimal $n \in \mathbb{N}$ such that there exist an A-module Z and a morphism $\varphi \in \operatorname{rad}_A^n(Z, X) \setminus \operatorname{rad}_A^{n+1}(Z, X)$ such that $f \circ \varphi \in \operatorname{rad}_A^{n+2}(Z, Y)$. Moreover, we put $d_l(f) := \infty$ if there is no such n. We define the right degree $d_r(f)$ of f dually.

We list properties of the left degree:

(1) If $f : X \to Y$ is an irreducible monomorphism (in particular, if Y is projective), then $d_l(f) = \infty$.

(2) If $f : X \to Y$ is an irreducible homomorphism, then $d_l(f) = 1$ if and only if the sequence $0 \to \operatorname{Ker} f \to X \to Y \to 0$ is almost split.

(3) If $f : X \to Y$ is an irreducible homomorphism and $i : Y \to Z$ is a projection, then $d_l(i \circ f) \leq d_l(f)$. Moreover, if $d_l(f) < \infty$ and $\dim_k Z < \dim_k Y$, then the inequality is strict.

(4) If
$$0 \to X \xrightarrow{\left[\begin{smallmatrix} g \\ g' \end{smallmatrix}\right]} Y \oplus Y' \xrightarrow{[f\ f']} Z \to 0$$
is an almost split sequence and $g \neq 0 \neq f'$, then $d_l(g) < \infty$ if and only if $d_l(f') < \infty$. Moreover, if this is the case then $d_l(g') = d_l(f) - 1$.

(5) If $f : X \xrightarrow{\left[\begin{smallmatrix} f' \\ f'' \end{smallmatrix}\right]} Y' \oplus Y''$ is an irreducible homomorphism and $d_l(f) < \infty$, then $d_l(f) = d_l(f') + d_l(f'')$.

We have the following improvement of the original definition.

Theorem (Chaio/Coelho/Trepode)**.** *If $f : X \to Y$ is an irreducible homomorphism between indecomposable modules, then $d_l(f)$ is the minimal $n \in \mathbb{N}$ such that there exist an A-module Z and a homomorphism $\varphi \in \operatorname{rad}_A^n(Z, X) \setminus \operatorname{rad}_A^{n+1}(Z, X)$ such that $f \circ \varphi = 0$.*

As a consequence we obtain the following.

Proposition (Chaio/Le Meur/Trepode)**.** *If $f : X \to Y$ is an irreducible homomorphism between indecomposable modules, then $d_l(f)$ is the minimal $n \in \mathbb{N}$ such that the inclusion map $\operatorname{Ker} f \hookrightarrow X$ belongs to $\operatorname{rad}_A^n(\operatorname{Ker} f, X)$.*

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Liu has proved that A is of finite representation type if and only if every homomorphism in $\operatorname{mod} A$ is a linear combination of compositions of irreducible morphisms. It is an open question if A is of finite representation type if and only if its Auslander–Reiten quiver is connected. We have the following characterization of the algebras of finite representation type.

Theorem (Chaio/Le Meur/Trepode). *The following conditions are equivalent.*

(1) *A is of finite representation type.*
(2) *If I is an indecomposable injective module and $\pi_I : I \to I/\operatorname{soc} I$ is the canonical projection, the $d_l(\pi_I) < \infty$.*
(3) *If P is an indecomposable projective module and $\iota_I : \operatorname{rad} P \to P$ is the canonical injection, the $d_r(\iota_P) < \infty$.*
(4) *If f is an irreducible epimorphism, then $d_l(f) < \infty$.*
(5) *If f is an irreducible monomorphism, then $d_r(f) < \infty$.*

Recall that Auslander has proved that A is of finite representation type if and only if $\operatorname{rad}^\infty(\operatorname{mod} A) = 0$ or, equivalently, there exists $n \in \mathbb{N}$ such that $\operatorname{rad}^n(\operatorname{mod} A) = 0$. It is an interesting problem to find, assuming that A is of finite representation type, the minimal $n \in \mathbb{N}$ such that $\operatorname{rad}^n(\operatorname{mod} A) = 0$. From the Harada–Sai lemma it follows that $\operatorname{rad}^{2^b-1}(\operatorname{mod} A) = 0$, where b is the maximum of the dimensions of the indecomposable A-modules. Eisenbud and de la Peña have improved this results as follows. For each simple module S denote by $P(S)$ and $I(S)$ its projective cover and injective envelope, respectively. Let

$$l := \max\{\min\{\dim_k P(S), \dim_k I(S)\} : S \text{ is a simple module}\}.$$

Then $\operatorname{rad}^{2^b-2^{l-1}+2}(\operatorname{mod} A) = 0$.

We have also the following.

Theorem. *If A is of finite representation type and*

$$m := \max\{d_r(\iota_{P(S)}) + d_l(\pi_{I(S)}); S \textit{ is a simple module}\},$$

then $\operatorname{rad}^m(\operatorname{mod} A) = 0$ *and* $\operatorname{rad}^{m+1}(\operatorname{mod} A) \neq 0$.