SHORT CHAINS AND SHORT CYCLES OF MODULES

BASED ON THE TALK BY ALICJA JAWORSKA

Throughout the talk A is an artin algebra over a commutative artin ring R.

1. Short chains and short cycles of modules

We say that an indecomposable A-module M lies on a short cycle if there exists an indecomposable A-module N such that

$$\operatorname{rad}_A(M,N) \neq 0 \neq \operatorname{rad}_A(N,M).$$

For an A-module M we denote by $[M]$ its image in the Grothendieck group of A. The aim of this section is to prove the following theorem.

Theorem 1.1 (Reiten/Skowroński/Smalø). *Let M and N be indecomposable A-modules such $[M] = [N]$. If M does not lie on a short cycle, then $M \simeq N$.*

We will need the following classical lemma. For A-modules M and N we denote by $[M, N]$ the length of the R-module $\operatorname{Hom}_A(M, N)$.

Lemma 1.2 (Auslander/Reiten). *Let X and Z be A-modules.*

(1) *If $P_1 \to P_0 \to X \to 0$ is a minimal projective presentation of X, then*

$$[X,Z] - [Z,\tau X] = [P_0,Z] - [P_1,Z].$$

(2) *If $0 \to X \to I_0 \to I_1$ is a minimal injective presentation of X, then*

$$[Z,X] - [\tau^- X,Z] = [Z,I_0] - [Z,I_1].$$

As an immediate consequence we obtain the following.

Corollary 1.3. *Let M and N be A-modules. If $[M] = [N]$, then*

$$[X,M] - [M,\tau X] = [X,N] - [N,\tau X]$$

and

$$[M,X] - [\tau^- X,M] = [N,X] - [\tau^- X,N]$$

for each A-module X.

Date: 24.03.2011.

We say that an indecomposable A-module M is the middle of a short chain if there exists an indecomposable A-module X such that

$$\mathrm{Hom}_A(X, M) \neq 0 \neq \mathrm{Hom}_A(M, \tau X).$$

The following fact plays a crucial role in the proof of Theorem 1.1.

Proposition 1.4. *If M is an indecomposable A-module, then M lies on a short cycle if and only if M is the middle of a short chain.*

Proof. Part I. Assume that M is the middle of a short chain, and fix an indecomposable A-module X and non-zero homomorphisms $f : X \to M$ and $g : M \to \tau X$. Let

$$0 \to \tau X \xrightarrow{\alpha} E \xrightarrow{\beta} X \to 0$$

be an almost split sequence. Since α is a monomorphism, there exists an indecomposable direct summand E' of E such that $\pi \circ \alpha \circ g \neq 0$, where $\pi : E \to E'$ is the canonical injection. Let $\iota : E' \to E$ be the canonical injection.

If $\beta \circ \iota$ is an epimorphism, then $f \circ \beta \circ \iota \neq 0$. Consequently,

$$\mathrm{rad}_A(M, E') \neq 0 \neq \mathrm{rad}_A(E', M)$$

in this case.

Now assume that $\beta \circ \iota$ is a monomorphism. Then

$$h := \beta \circ \iota \circ \pi \circ \alpha \circ g \neq 0.$$

In particular, $\mathrm{rad}_A(M, X) \neq 0$. If f is not an isomorphism, then we immediately have $\mathrm{rad}_A(X, M) \neq 0$ and the claim follows. If f is an isomorphism, then $f \circ h \circ f \neq 0$, hence again $\mathrm{rad}_A(X, M) \neq 0$. ☐

For the converse implication we need the following lemma.

Lemma 1.5 (Happel/Ringel)**.** *Let X, Y and Z be indecomposable A-modules. If there exist non-zero homomorphisms $f : X \to Y$ and $g : Y \to Z$ such that $g \circ f = 0$, then there exists an indecomposable A-module W such that*

$$\mathrm{Hom}_A(X, \tau W) \neq 0 \neq \mathrm{Hom}_A(W, Z).$$

Proof. Let $C := \mathrm{Coker}\, f$ and $p : Y \to C$ be the canonical projection. There exists a homomorphism $g' : C \to Z$ such that $g = g' \circ p$. Moreover, $g' \neq 0$, hence there exists an indecomposable direct summand W of C such that $g' \circ \iota \neq 0$, where $\iota : W \to C$ is the canonical inclusion. Note that $\pi \circ p$ does not split, where $\pi : C \to W$ is the canonical projection, since Y is indecomposable. In particular, W is not projective. We show that $\mathrm{Hom}_A(X, \tau W) \neq 0$.

Let

$$0 \to \tau W \xrightarrow{\alpha} E \xrightarrow{\beta} W \to 0$$

be an almost split sequence. Since $\pi \circ p$ does not split, there exists a homomorphism $h : Y \to E$ such that $\beta \circ h = \pi \circ p$. Next, h induces a homomorphism $h' : X \to \tau W$ such that $\alpha \circ h' = h \circ f$. We show that $h' \neq 0$. Indeed, if $h' = 0$, then $h \circ f = 0$. Consequently, there exists $\gamma : C \to E$ such that $h = \gamma \circ p$. Note that

$$\pi \circ p = \beta \circ h = \beta \circ \gamma \circ p,$$

hence $\pi = \beta \circ \gamma$. Consequently,

$$\mathrm{Id}_W = \pi \circ \iota = \beta \circ \gamma \circ \iota,$$

where $\iota : W \to C$ be the canonical inclusion. This leads to a contradiction, since β is not a split epimorphism. $\square$

Proof of Proposition 1.4. Part II. Assume that M lies on a short cycle, and fix an indecomposable A-module N and non-zero radical homomorphisms $f : M \to N$ and $g : N \to M$. If $g \circ f = 0$, then Lemma 1.5 implies that there exists an indecomposable A-module W such that

$$\mathrm{Hom}_A(M, \tau W) \neq 0 \neq \mathrm{Hom}_A(W, M).$$

On the other hand, if $g \circ f \neq 0$, then there exists $t \in \mathbb{N}_+$ such that $(g \circ f)^t \neq 0$ and $(g \circ f)^{t+1} = 0$. Then the claim follows from an application of Lemma 1.5 for the morphisms $(g \circ f)^t$ and $g \circ f$. $\square$

Proof of Theorem 1.1. Corollary 1.3 implies that

$$[M, M] - [M, \tau M] = [M, N] - [N, \tau M].$$

Proposition 1.4 implies that M is not the middle of short chain, hence in particular $[M, \tau M] = 0$. Consequently, $[M, N] \neq 0$. Dually $[N, M] \neq 0$. Since M does not lie on a short cycle, $\mathrm{rad}_A(M, N) = 0$ or $\mathrm{rad}_A(N, M) = 0$. This implies that M and N are isomorphic. $\square$