VARIATIONS ON A THEOREM OF ORLOV

BASED ON THE TALK BY RAGNAR-OLAF BUCHWEITZ

Throughout the talk we assume that K is a field and A is a noetherian finitely generated non-negatively $\mathbb{Z}$-graded K-algebra such that $A_0 = K$. We also assume that the injective dimension of A both as a left and as a right module is finite. If this is the case, then they are equal and we denote this common value by d. Finally, we assume that there exists an integer a such that

$$\operatorname{Ext}^i_A(K, A) = \begin{cases} K(a) & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$$

for each $i \in \mathbb{N}$, where all Ext's we consider are Ext's in the category of graded modules. We call a the Gorenstein invariant of A.

We denote by $\operatorname{Modgr} A$ the category of graded A-modules and by $\operatorname{modgr} A$ the full subcategory of $\operatorname{Modgr} A$ consisting of the finitely generated ones. Next, we denote by $\operatorname{Tors} A$ the full subcategory of $\operatorname{Modgr} A$ consisting of the modules M such that for each $m \in M$ there exists $i \in \mathbb{N}$ with $m \cdot A_{\geq i} = 0$. Finally, we denote by $\operatorname{tors} A$ the intersection of $\operatorname{Tors} A$ with $\operatorname{modgr} A$, which consists of the modules of finite length. Then $\operatorname{tors} A$ is a Serre subcategory of $\operatorname{modgr} A$ and, in analogy to the projective geometry, the quotient category $\operatorname{modgr} A / \operatorname{tors} A$ can be viewed as the category $\operatorname{Coh} \mathbb{X}$ of coherent sheaves over some "scheme" $\mathbb{X}$.

Let $\mathbf{a} : \operatorname{Modgr} A \to \operatorname{Modgr} A / \operatorname{Tors} A$ be the projection functor. Then $\mathbf{a}$ has the right adjoint $\Gamma_* : \operatorname{Modgr} A / \operatorname{Tors} A \to \operatorname{Modgr} A$ such that

$$(\Gamma_* \circ \mathbf{a})(M) = \varinjlim_{i \in \mathbb{N}} \operatorname{Hom}_A(A_{\geq i}, M)$$

for each $M \in \operatorname{Modgr} A$. In general, $(\Gamma_* \circ \mathbf{a})(M)$ does not have to be finitely generated even if M is, however the composition $\Gamma_{\geq 0}$ of Γ_* with the restriction to non-negative degrees preserves finite generation. Consequently, we obtain a pair $(\mathbf{a}, R\Gamma_{\geq 0})$ of adjoint functors between the category $\mathcal{D}^b(\operatorname{modgr}_{\geq 0} A)$, which we denote shortly by $\mathcal{D}^b(A)$, and the category $\mathcal{D}^b(\operatorname{Coh} \mathbb{X})$.

Let $\mathcal{D}^b_{\mathrm{sg}}(A)$ be the singularity category of A, i.e. the quotient of the category $\mathcal{D}^b(A)$ by the category of perfect complexes. One shows that the projection functor $\mathbf{M}$ has the left adjoint, which we denote by $\mathbf{b}$. In this way we obtain a pair $(\mathbf{a} \circ \mathbf{b}, \mathbf{M} \circ R\Gamma_{\geq 0})$ of adjoint functors

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between the categories $\mathcal{D}^b_{\mathrm{sg}}(A)$ and $\mathcal{D}^b(\mathrm{Coh}\,\mathbb{X})$. We recall that $\mathcal{D}^b_{\mathrm{sg}}(A)$ is equivalent to the stable category of maximal Cohen–Macaulay graded modules, where a module M is called Cohen–Macaulay if

$$\mathrm{Ext}^i_A(M, A) = 0$$

for each $i \in \mathbb{N}$ such that $i \neq 0$. Another description of $\mathcal{D}^b_{\mathrm{sg}}(A)$ is via the homotopy category of the exact complexes of finitely generated projective A-modules.

Orlov's Theorem says that $\mathbf{a} \circ \mathbf{b}$ is essentially surjective with the kernel generated by K, $K(1)$, ..., $K(a-1)$if $a \geq 0$, while $\mathbf{M} \circ R\Gamma_{\geq 0}$ is essentially surjective witj ther kernel generated by $\mathcal{O}$, $\mathcal{O}(1)$, ..., $\mathcal{O}(-a+1)$ if $a \leq 0$, where $\mathcal{O} := \mathbf{a}(A)$. In particular, they are both equivalences if $a = 0$.