THE CONNECTION BETWEEN BRAID GROUP ACTIONS ON EXCEPTIONAL SEQUENCES, THICK SUBCATEGORIES, AND NON-CROSSING PARTITIONS

BASED ON THE TALK BY CLAUDIA KÖHLER

Let H be a finite dimensional hereditary algebra over a field k. We define the Euler bilinear form on the Grothendieck group $K_0(H)$ by the condition:

$$\langle [V], [W] \rangle = \dim_k \operatorname{Hom}_H(V, W) - \dim_k \operatorname{Ext}^1_H(V, W)$$

for H-modules V and W. Next, for $\alpha, \beta \in K_0(H)$ we put

$$(\alpha, \beta) := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle.$$

Finally, for $\beta \in K_0(H)$ we define $s_\beta \in \operatorname{GL}(K_0(H))$ by the formula:

$$s_\beta(\alpha) := \alpha - (\alpha, \beta) \cdot \beta$$

for $\alpha \in K_0(H)$. If $\{S_1, \ldots, S_n\}$ is a complete set of pairwise non-isomorphic simple H-modules, then we denote by W_H the subgroup of $\operatorname{GL}(K_0(H))$ generated by the maps $s_{[S_1]}$, ..., $s_{[S_n]}$. An element t of W_H is called a reflection if there exist $w \in W_H$ and $i \in \{1, \ldots, n\}$ such that $t = w \cdot s_{[S_i]} \cdot w^{-1}$. For $w \in W_H$ we define its absolute length $\ell(w)$ as the minimal $r \in \mathbb{N}$ such that $w = x_1 \cdot \ldots \cdot x_r$ for some reflections x_1, ..., x_r. Let $c := s_{[S_1]} \cdot \ldots \cdot s_{[S_n]}$. We call c a Coxeter element. One shows that $\ell(c) = n$. An element w of W_H is called a non-crossing partition if $\ell(w) + \ell(w^{-1} \cdot c) = \ell(c)$.

For example, let H be the path algebra of the quiver

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n}{\bullet}\ .$$

For $i \in \{1, \ldots, n\}$ let S_i be the simple module at vertex i. Then W_H is isomorphic to $\mathfrak{S}_{n+1}$ via the isomorphism which sens the reflection $s_{[S_i]}$ to the transposition $(i, i+1)$ for each $i \in \{1, \ldots, n\}$. We want to describe the non-crossing partitions in W_H with respect to $c := s_{[S_1]} \cdot \ldots \cdot s_{[S_n]}$. First, for each subset I of $\{1, \ldots, n+1\}$ we denote by σ_I the permutation $(i_1, \ldots, i_t)$ provided $I = \{i_1, \ldots, i_t\}$ and $i_1 < \ldots < i_t$. We call two subsets I and J of $\{1, \ldots, n+1\}$ non-crossing if there are no $i_1, i_2, i_3, i_4 \in \{1, \ldots, n+1\}$ such that $i_1 < i_2 < i_3 < i_4$ and either $i_1, i_3 \in I$ and $i_2, i_4 \in J$ or vice versa. If $\mathcal{N}$ is the set of all partitions of the set $\{1, \ldots, n+1\}$ into pairwise non-crossing subsets, then the map which assigns to a partition $\{I_1, \ldots, I_s\}$ the permutation $\sigma_{I_1} \cdot \ldots \cdot \sigma_{I_s}$

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is a bijection between $\mathcal{N}$ and the set of non-crossing partitions in W_H with respect to c.

Let again H be an arbitrary finite dimensional hereditary algebra over a field k. A full subcategory $\mathcal{S}$ of $\operatorname{mod} H$ is called thick if $\mathcal{S}$ is closed under direct summands and $X_1, X_2, X_3 \in \mathcal{S}$ for each exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ such that $|\{i \in \{1,2,3\} : X_i \in \mathcal{S}\}| \geq 2$. A sequence $(E_1, \ldots, E_r)$ of indecomposable H-modules is called exceptional if $\operatorname{Ext}^1_H(E_i, E_i) = 0$ for each $i \in \{1, \ldots, n\}$ and $\operatorname{Hom}_H(E_j, E_i) = 0 = \operatorname{Ext}^1_H(E_j, E_i)$ for all $i, j \in \{1, \ldots, n\}$ such that $i < j$. We call the sequence $(E_1, \ldots, E_r)$ complete if r equals the number of the isomorphism classes of the simple H-modules. For an exceptional sequence $(E_1, \ldots, E_r)$ we denote by $\operatorname{Thick}(E_1, \ldots, E_r)$ the smallest thick subcategory of $\operatorname{mod} H$ containing the modules $E_1, \ldots, E_r$.

Theorem (Igusa/Schiffler, Ingalls/Thomas). *Let H be the path algebra of an acyclic quiver and let $\{S_1, \ldots, S_n\}$ be a complete set of pairwise non-isomorphic simple H-modules. If $(S_1, \ldots, S_n)$ is an exceptional sequence and $c := s_{[S_1]} \cdot \ldots \cdot s_{[S_n]}$, then the assignment*

$$(E_1, \ldots, E_r) \mapsto s_{[E_1]} \cdot \ldots \cdot s_{[E_r]}$$

induces a bijection between the thick subcategories generated by exceptional sequences and the non-crossing partitions in W_H with respect to c.

For a positive integer n we denote by B_n be the braid group on n strands, i.e. the group generated by elements $\sigma_1, \ldots, \sigma_{n-1}$ such that $\sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i$ for all $i, j \in \{1, \ldots, n-1\}$ with $|i - j| > 1$ and $\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}$ for each $i \in \{1, \ldots, n-2\}$. Now let H be a finite dimensional hereditary algebra over a field k with n isomorphism classes of the simple H-modules. We fix a Coxeter element c in W_H. We have the action of B_n on the set $\mathcal{X}$ of all sequences $(x_1, \ldots, x_n)$ of reflections in W_H such that $x_1 \cdot \ldots \cdot x_n = c$ defined by the following:

$$\sigma_i \cdot (x_1, \ldots, x_n) := (x_1, \ldots, x_{i-1}, x_i x_{i+1} x_i^{-1}, x_i, x_{i+2}, \ldots, x_n)$$

for $i \in \{1, \ldots, n-1\}$ and $(x_1, \ldots, x_n) \in \mathcal{X}$. Moreover, we also want to define an action of B_n on the set $\mathcal{E}$ of the complete exceptional sequences. First, one shows that if $(E_1, \ldots, E_n) \in \mathcal{E}$ and $i \in \{1, \ldots, n-1\}$, then there exists unique indecomposable H-module X such that $(E_1, \ldots, E_{i-1}, X, E_i, E_{i+2}, \ldots, E_n) \in \mathcal{E}$. In the above situation we put

$$\sigma_i \cdot (E_1, \ldots, E_n) := (E_1, \ldots, E_{i-1}, X, E_i, E_{i+2}, \ldots, E_n),$$

and this definition induces the action of B_n on $\mathcal{E}$.

Theorem (Crawley-Boevey, Ringel, Igusa/Schiffler). *Let H be be a finite dimensional hereditary algebra over a field k and let $\{S_1, \ldots, S_n\}$ be a complete set of pairwise non-isomorphic simple H-modules. If*

$(S_1,\dots,S_n)$ is an exceptional sequence and $c := s_{[S_1]}\cdot\ldots\cdot s_{[S_n]}$, then the above described actions of B_n on $\mathcal{X}$ and $\mathcal{E}$ are transitive. Moreover, if $\sigma\in B_n$, $(E_1,\dots,E_n)\in\mathcal{E}$ and $(E_1',\dots,E_n') := \sigma\cdot(E_1,\dots,E_n)$, then

$$\sigma(s_{[E_1]},\dots,s_{[E_n]}) = (s_{[E_1']},\dots,s_{[E_n']}).$$

Using the above theorem we may find the thick subcategory corresponding to a given non-crossing partition w. Namely, if $\ell(w)=r$, then there exist reflections $x_1,\dots,x_r\in W_H$ such that $w = x_1\cdot\ldots\cdot x_r$. Since w is a non-crossing partition, there exist reflections $x_{r+1},\dots,x_n\in W_H$ such that $x_1\cdot\ldots\cdot x_n = c$. According to the above theorem we may find an element $\sigma\in B_n$ such that $\sigma\cdot(s_{[S_1]},\dots,s_{[S_n]}) = (x_1,\dots,x_n)$. If $(E_1,\dots,E_n) := \sigma\cdot(S_1,\dots,S_n)$, then w corresponds to the category $\mathrm{Thick}(E_1,\dots,E_r)$