CLUSTER THEORY AND LUSZTIG'S CANONICAL BASIS

BASED ON THE TALK BY PHILIPP LAMPE

Let $\mathfrak{g}$ be a simple complex Lie algebra. It is known that $\mathfrak{g}$ admits a triangular composition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_.$$

Moreover, the representations of the algebra $\mathfrak{g}$ correspond to the modules over the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the algebra $\mathfrak{g}$. The algebras $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{n})$ admit several Poincaré–Birkhoff–Witt bases. On the other hand, Lusztig has introduced a concept of the canonical base, where a base B is called canonical, if for each irreducible representation V the set $\{b \cdot v_0 : b \in B\} \setminus \{0\}$ is a basis of the representation V, where v_0 is the lowest weight vector in the representation V. Finally, it is known that the graded dual $\mathcal{U}(\mathfrak{n})^*_{\mathrm{gr}}$ is isomorphic to the coordinate ring of the unipotent group corresponding to the algebra $\mathfrak{n}_+$.

Fomin and Zelevisky have introduced cluster algebrasS in order to deal with the dual canonical basis. They have proved that every cluster variable is a Laurent polynomial with integer coefficients in the variables occuring in the initial cluster. Geiss, Leclerc and Schröer have proved that the cluster variables are irreducible, however in general a cluster algebra is not factorial. They have also proved that the algebra $\mathcal{U}(\mathfrak{n})^*_{\mathrm{gr}}$ carries a structure of a cluster algebra. One of the most important questions of the cluster theory is if every cluster variable (or maybe even, monomial) belongs to the dual canonical basis.

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