MONOID ALGEBRAS OF PROJECTION FUNCTORS

BASED ON THE TALK BY ANNA-LOUISE PAASCH

Throughout the talk we assume that A is a finite dimensional algebra over a field k.

For a simple A-module S we define the functor $P_S : \operatorname{mod} A \to \operatorname{mod} A$ by the formula:

$$P_S(M) := M / \sum_{f \in \operatorname{Hom}_A(S,M)} \operatorname{Im} f.$$

Obviously the image of the functor P_S contains the kernel of the functor $\operatorname{Hom}_A(S, -)$. Moreover, if $\operatorname{Ext}^1_A(S, S) = 0$, then we an equality. By Π_A we denote the monoid generated by the functors P_S, where S runs through the simple A-modules. It is an interesting problem to investigate the structure of the monoid Π_A.

One easily verifies that if S a simple A-module with $\operatorname{Ext}^1_A(S, S) = 0$, then $P_S^2 \sim P_S$. Moreover, if S and T are simple A-modules such that $\operatorname{Ext}^1_A(S, S) = 0 = \operatorname{Ext}^1_A(T, T)$ and $\operatorname{Ext}^1_A(T, S) = 0$, then

$$P_S \circ P_T \circ P_S \sim P_T \circ P_S \sim P_T \circ P_S \circ P_T.$$

For the rest of the talk we assume that A is the path algebra of a quiver Q. Motivated by the above observations we define the algebra $\mathcal{B}_Q$ as the algebra with the generators X_i, $i \in Q_0$, such that $X_i^2 = X_i$ for each vertex $i \in Q_0$, if i and j are vertices of the quiver Q joined by an arrow, then

$$X_i \cdot X_j \cdot X_i = X_j \cdot X_i = X_j \cdot X_i \cdot X_j,$$

and if i and j are vertices of the quiver Q, which are not joined by an arrow, then $X_i \cdot X_j = X_j \cdot X_i$. If the canonical morphism $\mathcal{B}_Q \to k\Pi_A$ is an isomorphism, then the above described relations define the monoid Π_A. This fact has been confirmed for some classes of quivers.

In the rest of the talk we will study the algebra $\mathcal{B}_Q$. One may check that the algebra $\mathcal{B}_Q$ is basic and finite dimensional. The simple $\mathcal{B}_Q$-modules corresponds to the subsets of the set of vertices of the quiver Q. We denote the simple $\mathcal{B}_Q$-modules corresponding to a subset M by E_M. If M and N are two such sets, then $\dim_k \operatorname{Ext}^1_{\mathcal{B}_Q}(E_M, E_N) \leq 1$ and $\operatorname{Ext}^1_{\mathcal{B}_Q}(E_M, E_N) \neq 0$ if and only if $M \setminus N \neq \varnothing \neq N \setminus M$ and each vertex from the set $M \setminus N$ is connected in the quiver Q with each vertex from the set $N \setminus M$. In particular, this implies that if Q is not an equioriented quiver of type $\mathbb{A}$, then the Gabriel quiver of the

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algebra $\mathcal{B}_Q$ has exactly 3 connected components. On the other hand, if Q is an equioriented quiver of type $\mathbb{A}$ with n vertices, then it has $n+1$ connected components. Moreover, in the latter case the algebra $\mathcal{B}_Q$ is an incidence algebra. More precisely, we we denote by $\mathcal{P}_n$ the partial set consisting of the subsets of the set $\{1,\dots,n\}$ with the order relation defined by: if I and J are subsets of the set $\{1,\dots,n\}$, then $I \leq J$ if and only if $|I| = |J|$ and $i_l < j_l$ for each $l \in \{1,\dots,|I|\}$, where $I = \{i_1 < \cdots < i_{|I|}\}$ and $J = \{j_1 < \cdots < j_{|J|}\}$. Then the incidence algebra of the poset $\mathcal{P}_n$ is isomorphic to the algebra $\mathcal{B}_Q$.