AN INTRODUCTION TO THE ZIEGLER SPECTRUM

BASED ON THE TALK BY GENA PUNINSKI

Throughout the talk all algebras are finite-dimensional. The aim of this talk is to define, for an algebra A, the Ziegler spectrum Zg_A. It is a topological space whose points are the isomorphism classes of the indecomposable pure-injective left A-modules. Thus we begin with the definition of a pure-injective module.

We say that a module M is pure-injective if M is a direct summand of the product of finite dimensional modules. From the definition it immediately follows that the finite dimensional modules are pure-injective. Moreover, the class of pure-injective modules is closed under direct products and direct summands.

The pure-injective modules can be characterized in a different way, which explains the name. We say that monomorphism f of left modules is pure, if $f \otimes_A K$ is a monomorphism for each right A-module K. For example, if $f : M \to N$ is a monomorphism and M is finite dimensional, then f is pure if and only if f splits. One shows that a module M is pure-injective if and only if every pure monomorphism $f : M \to N$ splits. This immediately implies that the injective modules are pure-injective. Moreover, the endofinite modules are pure-injective. Finally, every module which is linearly compact over its endomorphism ring is pure-injective. Recall, that a module M is called linearly compact if for all submodules M_i, $i \in I$, of M and elements x_i, $i \in I$, of M there exists $x \in M$ such that $x - x_i \in M_i$ for all $i \in I$ if and only if for each finite subset J of I there exists $x_J \in M$ such that $x_J - x_i \in M_i$ for all $i \in J$.

The other characterization of the pure-injective modules is the following. A module M is pure injective if and only if for each set I there exists homomorphism $f : M^I \to M$ such that $f((m_i)) = \sum_{i \in I} m_i$ for each $(m_i) \in M^{(I)}$. This characterization implies immediately that if M is a pure-injective A-module and $B \to A$ is a homomorphism of algebras, then M is a pure-injective B-module.

For each module M we may define its pure-injective envelope, i.e. a pure monomorphism $f : M \to N$ such that N is pure-injective and for each pure monomorphism $g : M \to L$ with L pure-injective there exists unique homomorphism $h : N \to L$ such that $g = h \circ f$. Existence of pure-injective envelopes can be proved in the following way. Let A

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be an algebra. There is the embedding of the category of the left A-modules into the category of the functors from the category of the finite dimensional right modules to the category of the abelian groups, which sends a module M to the functor $- \otimes_A M$. One knows that $\mathrm{Hom}(- \otimes_A M, F) \simeq \mathrm{Hom}(M, F(A))$. Next, the sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ is pure-exact (i.e. f is a pure monomorphism) if and only if the sequence

$$0 \to - \otimes_A M \xrightarrow{- \otimes_A f} - \otimes_A N \xrightarrow{- \otimes_A g} L \to 0$$

is exact. Finally, a module M is pure-injective if and only if the functor $- \otimes_A F$ is injective and all injective functors are of this form. Consequently, if $- \otimes_A f : - \otimes_A M \to - \otimes_A N$ is an injective envelope of the functor $- \otimes_A M$, then $f : M \to N$ is a pure-injective envelope of M.

Now we come to the definition of the Zeigler spectrum Zg_A of an algebra A. Recall that the elements of Zg_A are the isomorphism classes of the indecomposable pure-injective A-modules. Next, for each homomorphism $f : M \to N$ between finite dimensional modules we denote by (f) the subset of Zg_A consisting of the isomorphism classes of the modules L such that $\mathrm{Im}\,\mathrm{Hom}_A(f, L) \neq \mathrm{Hom}_A(M, L)$. Ziegler has proved that the sets of the above form a basis for a quasi-compact topology in Zg_A. Prest has observed that the isomorphism classes of the indecomposable finite dimensional modules are the isolated points of Zg_A. Moreover, the set $\mathrm{ind}\,A$ of these isomorphism classes is dense in Zg_A. In particular, if A is not of finite representation type, then there exists an infinite dimensional indecomposable pure-injective module.

As an illustration we describe the Ziegler spectrum of the Kronecker algebra H. Recall, that in this case for each $\lambda \in \mathbb{P}^1(k)$ we have the corresponding adic module Q_λ and the corresponding Prüfer module P_λ. Moreover, we denote by G the generic module. Finally, for each $\lambda \in \mathbb{P}^1(k)$ we denote by R_λ the corresponding simple regular module. Then

$$\mathrm{Zg}_H = \mathrm{ind}\,H \cup \{[Q_\lambda], [P_\lambda] : \lambda \in \mathbb{P}^1(k)\} \cup \{[G]\}.$$

Moreover, a subset C of Zg_H is closed if and only if the following conditions are satisfied:

(1) If $\lambda \in \mathbb{P}^1(k)$ and
$$|C \cap \{[M] \in \mathrm{Zg}_H : \mathrm{Hom}(R_\lambda, M) \neq 0\}| = \infty,$$
then $[P_\lambda] \in C$.

(2) If $\lambda \in \mathbb{P}^1(k)$ and
$$|C \cap \{[M] \in \mathrm{Zg}_H : \mathrm{Hom}(M, R_\lambda) \neq 0\}| = \infty,$$
then $[Q_\lambda] \in C$.

(3) If $|C \cap \mathrm{ind}\,H| = \infty$ or $C \setminus \mathrm{ind}\,H \neq \varnothing$, then $[G] \in C$.