DIRECT SUMMANDS OF HOMOLOGICAL FUNCTORS

BASED ON THE TALK BY ALEX MARTSINKOVSKY

The aim of this talk is to present a proof (due to Auslander) of the following theorem.

Theorem. *Let $\mathcal{C}$ be an abelian category with enough projectives. Let F be a direct summand of the functor $\operatorname{Ext}^1_{\mathcal{C}}(C, -)$ for an object C of $\mathcal{C}$. If either $\mathcal{C}$ has countable direct sums or $\operatorname{pd}_{\mathcal{C}} C < \infty$, then there exists an object D of $\mathcal{C}$ such that*

$$F \simeq \operatorname{Ext}^1_{\mathcal{C}}(D, -).$$

The former part (if $\mathcal{C}$ has countable direct sums) of this theorem was first proved by Freyd, while the latter (if $\operatorname{pd}_{\mathcal{C}} C < \infty$) is due to Auslander.

We fix an abelian category $\mathcal{C}$. Let $\operatorname{Mod} \mathbb{Z}$ be the category of abelian groups. Recall that a functor $F : \mathcal{C} \to \operatorname{Mod} \mathbb{Z}$ is called finitely presented, if there exists a map $f : A \to B$ such that

$$C \simeq \operatorname{Coker}(f, -)_{\mathcal{C}} \simeq F.$$

Here, if $*$ is either an object or a morphism in $\mathcal{C}$, then we put

$$(*, -)_{\mathcal{C}} := \operatorname{Hom}_{\mathcal{C}}(*, -) \qquad \text{and} \qquad (*, -)^1_{\mathcal{C}} := \operatorname{Ext}^1_{\mathcal{C}}(*, -).$$

We denote by $\check{\mathcal{C}}$ the category of finitely presented functors $\mathcal{C} \to \operatorname{Mod} \mathbb{Z}$. This is an abelian category. Moreover, by $\check{\mathcal{C}}_0$ we denote the full subcategory of $\check{\mathcal{C}}$ formed by these functors, for which we may choose f to be a monomorphism. The category $\check{\mathcal{C}}_0$ is again abelian. Note that $(A, -)^1_{\mathcal{C}}$ is an object of $\check{\mathcal{C}}_0$ for each object A of $\mathcal{C}$. Indeed, in $\mathcal{C}$ we have an exact sequence

$$0 \to \Omega A \to P \to A \to 0$$

with P projective, which induces a sequence

$$0 \to (A, -)_{\mathcal{C}} \to (P, -)_{\mathcal{C}} \to (\Omega A, -)_{\mathcal{C}} \to (A, -)^1_{\mathcal{C}} \to 0.$$

We have the following two easy lemmas.

Lemma. *If G is an object of $\check{\mathcal{C}}_0$, there there exists an object C of $\mathcal{C}$ such that G is a subobject of $(C, -)^1_{\mathcal{C}}$.*

Proof. By definition there exists a monomorphism $f : A \to B$ such that $\operatorname{Coker}(f, -)_{\mathcal{C}} \simeq G$. On the other hand, we have an exact sequence

$$(B, -)_{\mathcal{C}} \xrightarrow{(f,-)_{\mathcal{C}}} (A, -)_{\mathcal{C}} \to (\operatorname{Coker} f, -)^1_{\mathcal{C}},$$

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hence we get an inclusion

$$G \simeq \operatorname{Coker}(f,-)_{\mathcal{C}} \hookrightarrow (\operatorname{Coker} f,-)^1_{\mathcal{C}}. \qquad \square$$

Lemma. *A finitely presented functor $F : \mathcal{C} \to \operatorname{Mod}\mathbb{Z}$ is half exact if and only if $\operatorname{Ext}^1_{\check{\mathcal{C}}}(G,F) = 0$ for each functor G in $\check{\mathcal{C}}_0$.*

Proof. Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be an exact sequence in $\mathcal{C}$. If $G = \operatorname{Coker}(f,-)_{\mathcal{C}}$, then we have the following projective presentation

$$0 \to (C,-)_{\mathcal{C}} \to (B,-)_{\mathcal{C}} \to (A,-) \to G \to 0$$

in $\check{\mathcal{C}}$, hence the Yoneda lemma implies that

$$\operatorname{Ext}^1_{\check{\mathcal{C}}}(G,F) \simeq \operatorname{Im} Ff / \operatorname{Ker} Fg.$$

Now the claim follows. $\square$

The above two lemmas immediately imply the following.

Proposition. *The following are equivalent for a functor F in $\check{\mathcal{C}}_0$.*

1. *F is half exact.*
2. *$\operatorname{Ext}^1_{\check{\mathcal{C}}_0}(G,F) = 0$ for each functor G in $\check{\mathcal{C}}_0$.*
3. *F is injective in $\check{\mathcal{C}}_0$.*
4. *F is a direct summand of $(C,-)^1_{\mathcal{C}}$ for some object C of $\mathcal{C}$.*

Now we fix in addition an object C of $\mathcal{C}$ and a direct summand F of $\operatorname{Ext}^1_{\mathcal{C}}(C,-)$.

From the Yoneda lemma there exists an endomorphism f of C such that $F = \operatorname{Ker}(f,-)^1_{\mathcal{C}}$. If $g : P \to C$ is an epimorphism with P projective, then we get an exact sequence

$$0 \to A \to C \oplus P \xrightarrow{(g,f)} C \to 0,$$

which in turn induces the long exact sequence

$$\begin{aligned} 0 \to F \to (C,-)^1_{\mathcal{C}} \to (C,-)^1_{\mathcal{C}} \to (A,-)^1_{\mathcal{C}} \\ \to (\Omega C,-)^1_{\mathcal{C}} \to (\Omega C,-)^1_{\mathcal{C}} \to (\Omega A,-)^1_{\mathcal{C}} \to \cdots, \end{aligned}$$

where for an object B of $\mathcal{C}$ we denote by ΩB the kernel of a chosen (and fixed) epimorphism $Q \to B$ with Q projective (we also assume that $\Omega B = 0$ if B is projective). Since F is injective in $\check{\mathcal{C}}_0$ according to the above proposition, this sequences splits. Consequently,

$$F \oplus \prod_{i \geq 1} (\Omega^i C,-) \oplus \prod_{i \geq 0} (\Omega^{2i+1} A,-) \simeq \prod_{i \geq 1} (\Omega^i C,-) \oplus \prod_{i \geq 0} (\Omega^{2i} A,-).$$

If $\mathcal{C}$ has countable direct sums, then we get an isomorphism

$$F \oplus (M,-)^1_{\mathcal{C}} \simeq (N,-)^1_{\mathcal{C}},$$

where

$$M := \bigoplus_{i\geq 1} \Omega^i C \oplus \bigoplus_{i\geq 0} \Omega^{2i+1} A$$

and

$$N := \bigoplus_{i\geq 1} \Omega^i C \oplus \bigoplus_{i\geq 0} \Omega^{2i} A.$$

We get the same conclusion, if $\operatorname{pd}_{\mathcal{C}} C < \infty$, since then also $\operatorname{pd}_{\mathcal{C}} A < \infty$, and $\Omega^i C = 0 = \Omega^i A$ for $i \gg 0$.

Now we use the following result due to Hilton and Rees.

Theorem. *If $h : N \to M$ is a morphism in $\mathcal{C}$, then $(h, -)^1_{\mathcal{C}}$ is a monomorphism if and only if for each epimorphism $Q \to M$ with Q projective the induced epimorphism $N \oplus Q \to M$ splits.*

We have an inclusion

$$(M, -)^1_{\mathcal{C}} \to F \oplus (M, -)^1_{\mathcal{C}} \simeq (N, -)^1_{\mathcal{C}},$$

which is given by a morphism $h : N \to M$ in $\mathcal{C}$. Fix an epimorphism $Q \to M$ with P projective. The above theorem implies that we get a split exact sequence

$$0 \to D \to N \oplus Q \to M \to 0,$$

which implies that $F \simeq (D, -)^1_{\mathcal{C}}$. This finishes the proof of our theorem.

We present one more situation, where we have a similar conclusion.

Theorem. *Let $G : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \operatorname{Mod} \mathbb{Z}$ be a bifunctor, where $\mathcal{A}$ is an abelian category, such that for each object A of $\mathcal{A}$ the natural map*

$$(A, A)_{\mathcal{A}} \to \operatorname{Mor}(G(A, -), G(A, -))$$

is onto. If A is an object of finite length of $\mathcal{A}$ and F is a direct summand of $G(A, -)$, then there exists a subobject B of A such that

$$F \simeq G(B, -).$$

Proof. Let $\iota : F \to G(A, -)$ and $\pi : G(A, -) \to F$ be the canonical inclusion and projection (in particular, $\pi \circ \iota = \mathrm{Id}_F$). By our assumption there exists a morphism $f : A \to A$ such that

$$\iota \circ \pi = G(f, -).$$

If f is an epimorphism, then it is an isomorphism (since A has finite length) and $F \simeq G(A, -)$. Otherwise, we write $f = h \circ g$ for an epimorphism $g : A \to B$ and a monomorphism $h : B \to A$ (in particular, the length of B is smaller than the length of A). Note that

$$(\pi \circ G(g, -)) \circ (G(h, -) \circ \iota) = \pi \circ G(f, -) \circ \iota = \pi \circ \iota \circ \pi \circ \iota = \mathrm{Id}_F,$$

hence F is a direct summand of $G(B, -)$ and the claim follows by induction. □