QUIVERS WITH RELATIONS ASSOCIATED WITH CARTAN MATRICES

BASED ON THE TALK BY JAN SCHRÖER

The talk is based on joint results with Christof Geiss and Bernard Leclerc.

1. Cartan matrices

An $n \times n$ matrix C with integer coefficients is called a symmetrizable generalized Cartan matrix if

(1) $c_{ii} = 2$ for each i,
(2) $c_{ij} \leq 0$ for all i and j with $i \neq j$,
(3) there exists a diagonal matrix D, called a symmetrizer of C, with positive integer coefficients such that DC is symmetric.

Given a symmetrizable generalized Cartan matrix C by an orientation we mean a set $\Omega \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ such that $\Omega \cap \{(i, j), (j, i)\} \neq \varnothing$ if and only if $c_{ij} < 0$. If $(i, j) \in \Omega$, then we assume that we have an arrow $i \leftarrow j$. We call the orientation Ω acyclic if there are no oriented cycles in the quiver, which we obtain in this way.

Given a symmetrizable generalized Cartan matrix, a symmetrizer D, and an orientation Ω, we define the bilinear form $\langle -, - \rangle_{C,D,\Omega}$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle_{C,D,\Omega} := \sum_{i=1}^{n} d_i x_i y_i + \sum_{(j,i) \in \Omega} d_i c_{ij} x_i y_j.$$

2. Species

Assume we are given a symmetrizable generalized Cartan matrix C, a symmetrizer D, and an orientation Ω. Fix a field F and F-skew fields F_i, $i = 1, \ldots, n$, such that $\dim_F F_i = d_i$. Moreover, for each $(i, j) \in \Omega$, we fix an F_i-F_j-bimodule F_{ij} with a central action of F, which is free of rank $|c_{ij}|$ as a left F_i-module and free of rank $|c_{ji}|$ as a right F_j-module. Next we put

$$S := \prod_{i=1}^{n} F_i, \qquad B := \bigoplus_{(i,j) \in \Omega} F_{ij},$$

and

$$A := T_S(B) := \bigoplus_{s \geq 0} B^{\otimes_S s}.$$

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Note that an A-module M is given by a collection of F_i-vector spaces M_i, $i = 1, \dots, n$, and F_i-linear maps $M_{ij}\colon F_{ij} \otimes_{F_j} M_j \to M_i$, $(i,j) \in \Omega$.

Theorem (Dlab/Ringel)**.** (1) *The algebra A is hereditary.*
(2) *The algebra A is of finite representation type if and only if C is of Dynkin type.*

Moreover, if this is the case, then there is a bijection between the isomorphism classes of the indecomposable A-modules and the positive roots of the Lie algebra $\mathfrak{g}(C)$, which assigns to each indecomposable A-module its dimension vector.

3. Alternative approach

Again let C be a symmetrizable generalized Cartan matrix, D a symmetrizer, and Ω an orientation. Our aim in this section is to define an algebra $H = H(C, D, \Omega)$. We first define its quiver $Q = Q(C, \Omega)$. The set of vertices of Q is $\{1, \dots, n\}$. For each $i = 1, \dots, n$, we have a loop ε_i ar i. Moreover, for each $(i,j) \in \Omega$ we have g_{ij} arrows $j \to i$, where g_{ij} is the greatest common divisor of $|c_{ij}|$ and $|c_{ji}|$. The are two types of relations, which we put on Q. First, for each $i = 1, \dots, n$, $\varepsilon_i^{d_i} = 0$. In addition, $\varepsilon_i^{f_{ji}}\alpha = \alpha\varepsilon_j^{f_{ij}}$, for each arrow $\alpha\colon j \to i$ such that $d_i d_j \neq 1$, where $f_{ji} := \frac{|c_{ji}|}{g_{ij}}$ and $f_{ij} := \frac{|c_{ij}|}{g_{ij}}$. Note that if C is symmetric and D is the identity matrix, then $H = KQ^0$, where Q is the quiver obtained from Q by removing loops. More generally, if C is symmetric and D is m times the identity matrix, then $H = KQ^0 \otimes_K K[x]/(x^m)$.

4. Analogy

Let C be a symmetrizable generalized Cartan matrix, D a symmetrizer, Ω an orientation, and $H = H(C, D, \Omega)$. For each vertex i, $H_i := e_i H e_i = K[\varepsilon_i]/(\varepsilon_i^{d_i})$. Next, if $(i,j) \in \Omega$, then $H_{ij} := e_i H e_j$. This is an H_i-H_j-bimodule, which is free of rank $|c_{ij}|$ as a left H_i-module and free of rank $|c_{ji}|$ as a right H_j-module. An H-module M can be viewed as a collection of H_i-modules M_i, $i = 1, \dots, n$, and H_i-linear maps $M_{ij}\colon H_{ij} \otimes_{H_j} M_j \to M_i$, $(i,j) \in \Omega$. Using this description one defines, for a sink i, a reflection functor

$$S_i^+\colon \operatorname{mod} H(C, D, \Omega) \to \operatorname{mod} H(C, D, \sigma_i^+\Omega).$$

This functor induces an equivalence between the category of H-modules M, such that S_i is not a composition factor of $\operatorname{top} M$, and the category $H(C, D, \sigma_i^+\Omega)$-modules, such that S_i is not a composition functor of $\operatorname{soc} M$. Like in the classical setting we obtain in this way the Coxeter functor $C^+\colon \operatorname{mod} H \to \operatorname{mod} H$. We have the following.

Theorem (Geiss/Leclerc/Schröer)**.** *Up to twist the functor C^+ is isomorphic to the functor $D\operatorname{Ext}_H^1(-, H)$.*

5. Representation theory of H

We start with the following.

Proposition. *For an H-module M the following conditions are equivalent:*

(1) $\mathrm{pd}_H M \leq 1$.
(2) $\mathrm{pd}_H M < \infty$.
(3) $\mathrm{id}_H M \leq 1$.
(4) $\mathrm{id}_H M < \infty$.
(5) *M is locally free, i.e. $e_i M$ is a free H_i-module, for each i.*

We say that an H-module is τ-locally free if $\tau_H^k M$ is locally free, for each $k \in \mathbb{Z}$.

Theorem (Geiss/Leclerc/Schröer)**.** *There only finitely many isomorphism classes of indecomposable τ-locally free H-module if and only if C is of Dynkin type. Moreover, if this is the case, then there is a bijection between the isomorphism classes of the indecomposable A-modules and the positive roots of the Lie algebra $\mathfrak{g}(C)$, which assigns to each indecomposable A-module its rank vector.*

Theorem (Geiss/Leclerc/Schröer)**.** *Let C be of wild type and $\mathcal{C}$ be a connected component of the Auslander–Reiten quiver of H. If $M \in \mathcal{C}$ is τ-locally free and neither preprojective nor preinjective, then $\mathcal{C}$ is of type $\mathbb{Z}\mathbb{A}_\infty$ and all modules in $\mathcal{C}$ are τ-locally finite.*

Is it not known if the positive roots of the Lie algebra $\mathfrak{g}(C)$ coincide with the ranks of the indecomposable τ-locally free modules.

6. Singularity category of H

An H-module M is called Gorenstein projective if it is a submodule of a projective H-module or, equivalently, $\mathrm{Ext}^1_H(M, H) = 0$. We denote by $\mathrm{GP}(H)$ the subcategory of Gorenstein projective modules. It is a Frobenius category and the stable category $\underline{\mathrm{GP}}(H)$ is equivalent to the singularity category $\mathcal{D}^b(\mathrm{mod}\, H)/\mathcal{K}^b(\mathrm{proj}\, H)$, due to results of Buchweitz and Orlov.

Theorem (Ringel/Zhang)**.** *If C is symmetric and $d_i = 2$, for each i, then*

$$\underline{\mathrm{GP}}(H) \simeq \mathcal{D}^b(\mathrm{mod}\, KQ^0)/[1].$$

7. Varieties of locally free modules

Let $\mathbf{e}$ be a dimension vector such that $d_i \mid e_i$, for each i. We put $\mathbf{r} := (\frac{e_1}{r_1}, \dots, \frac{e_n}{r_n})$. By $\mathrm{rep}_{\mathrm{lf}}(H, \mathbf{r})$ we denote the open subset of $\mathrm{rep}(H, \mathbf{e})$ consisting of locally free representations with rank vector $\mathbf{r}$.

Proposition. *The variety of* $\mathrm{rep}_{\mathrm{lf}}(H, \mathbf{r})$ *is smooth and irreducible of dimension* $\dim G_{\mathbf{d}} - q(\mathbf{r})$*, where*

$$q(\mathbf{x}) := \langle \mathbf{x}, \mathbf{x} \rangle_{C,D,\Omega}.$$

For each rank vector $\mathbf{r}$, there exist uniquely determined rank vectors $\mathbf{r}_1, \ldots, \mathbf{r}_k$ and dense open subset $U \subseteq \mathrm{rep}_{\mathrm{lf}}(H, \mathbf{r})$ such that, for each $M \in \mathcal{U}$, there exist indecomposable modules M_i with rank vector $\mathbf{r}_i$ such that $M = M_1 \oplus \cdots \oplus M_k$.

Theorem (Geiss/Leclerc/Schröer)**.** *The canonical decomposition of* $\mathbf{r}$ *does not depend on* D*.*