RIGID MODULES AND SCHUR ROOTS

BASED ON THE TALK BY JAN SCHRÖER

The talk was based on joint results with Christof Geiß and Bernard Leclerc.

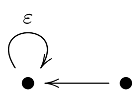
1. Introduction

Given a Cartan datum (C, D, Ω) one may associate to it a species $\tilde{H} = \tilde{H}(C, D, \Omega)$, which is a finite dimensional hereditary algebra over a given field K, which is defined as the tensor algebra $T_S(B)$, for some algebra S and an S-S-bimodule B. For example, for a Cartan datum of type $\mathbb{B}_2$, $S := \mathbb{C} \times \mathbb{R}$ and $B := {}_{\mathbb{C}}\mathbb{C}_{\mathbb{R}}$, thus

$$\tilde{H} = \begin{bmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} \end{bmatrix}.$$

It is known that in this case $\tilde{H}$ has 4 indecomposable representations, where a representation of $\tilde{H}$ consists of a $\mathbb{C}$-vector space M_1 and a $\mathbb{R}$-vector space M_2 together with a $\mathbb{C}$-linear map $B \otimes_{\mathbb{R}} M_2 \to M_1$. If C is symmetric, then one may take any ground field K, however if C is not symmetric, there are restrictions. In particular, K cannot be algebraically closed, thus there is no geometry behind this construction.

In 2016, Geiß, Leclerc and Schröer introduced, for any field K and every Cartan datum (C, D, Ω), a finite dimensional 1-Gorenstein K-algebra $H = H(C, D, \Omega)$ defined by a quiver with relations. These algebras are called generalized species. For example, in the above situation, H is the path algebra of the quiver



bound by the relation $\varepsilon^2 = 0$. There are exactly 4-indecomposable rigid representations of H, where a representation M is called rigid, if $\operatorname{Ext}^1(M, M) = 0$. The aim of the talk was to relate $\tilde{H}$ and H.

If (C, D, Ω) is a Cartan datum, such that C is symmetric and D is the identity matrix, then $\tilde{H}$ is the path algebra of the associated quiver. One has the following geometric version of Ringel's Hall algebra theorem.

Theorem (Schofield, 1990)**.** *Let (C, D, Ω) be a symmetric Cartan datum, such that C is symmetric and D is the identity matrix. Then the*

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convolution algebra $\mathcal{M}(\tilde{H})$ is isomorphic to the enveloping algebra of the positive part $\mathfrak{n}(C)$ of the Kac–Moody algebra associated to C.

The following theorem gives another motivation for studying H.

Theorem (Geiß/Leclerc/Schröer, 2017)**.** *If (C, D, Ω) is a Dynkin Cartan datum, where D is the identity matrix, then the convolution algebra $\mathcal{M}(H(C, D, \Omega))$ is isomorphic to the enveloping algebra of $\mathfrak{n}(C)$.*

Note that the above theorem covers the cases of Dynkin quivers of types $\mathbb{B}$, $\mathbb{C}$, $\mathbb{F}$ and $\mathbb{G}$.

2. Three classes of algebras

Throughout this section (C, D, Ω) is a Cartan datum. This means that C is a Cartan matrix, i.e. $C = (c_{i,j}) \in \mathbb{M}_n(\mathbb{Z})$, $c_{i,i} = 2$, for each i, and $c_{i,j} \leq 0$, for each $i \neq j$. Moreover, D is a diagonal matrix with positive integer coefficients c_i, $1 \leq i \leq n$, on the diagonal, such that the matrix DC is symmetric. Finally, Ω is a set of pairs (i, j), $1 \leq i, j \leq n$, such that Ω has a nonempty intersection with the set $\{(i, j), (j, i)\}$ if and only if $c_{i,j} < 0$. Moreover, one also assumes that there is no sequence $(i_0, i_1, \ldots, i_m)$ of integers, such that $m > 0$, $(i_{p-1}, i_p) \in \Omega$, for each $1 \leq p \leq m$, and $i_0 = i_p$. We denote by $g_{i,j}$ the greatest common divisor of $c_{i,j}$ and $c_{j,i}$, and put $f_{i,j} := \frac{|c_{i,j}|}{g_{i,j}}$, if $c_{i,j} < 0$.

2.1. **The first class of algebras.** We define now a quiver Q and an ideal $I \subseteq KQ$, such that $H = H(C, D, \Omega) := KQ/I$. First, $Q_0 := \{1, \ldots, n\}$. Next, for each pair $(i, j) \in \Omega$, we have arrows $\alpha_{i,j}^{(g)} \colon j \to i$, for $1 \leq g \leq g_{i,j}$. Moreover, for each vertex i we also have a loop ε_i at i. Finally, the ideal I is generated by relations:

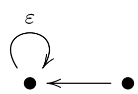
- $\varepsilon_i^{c_i} = 0$, for each $1 \leq i \leq n$;
- $\varepsilon_i^{f_{j,i}} \alpha_{i,j}^{(g)} = \alpha_{i,j}^{(g)} \varepsilon_j^{f_{i,j}}$, for each $(i, j) \in \Omega$ and $1 \leq g \leq g_{i,j}$.

We observe that the second class of relations appears in Cecotti's work on supersymmetry.

For example, if

$$C := \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}, \qquad D := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \Omega := \{(1, 2)\},$$

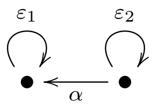
then we get the path algebra of the quiver



bound by the relation $\varepsilon^2 = 0$. On the other hand, if C and Ω are as above and

$$D := \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix},$$

then H is the path algebra of the quiver

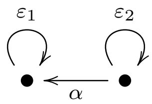


bound by the relations $\varepsilon_1^4 = 0$, $\varepsilon_2^2 = 0$ and $\varepsilon_1^2\alpha = \alpha\varepsilon_2$.

In the both examples above the Cartan datum was of type $\mathbb{B}_2$. If

$$C := \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad D := \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \text{and} \qquad \Omega := \{(1,2)\},$$

(in particular, the Cartan datum is of type $\mathbb{A}_2$), then H is the path algebra of the quiver



bound by the relations $\varepsilon_1^m = 0$, $\varepsilon_2^m = 0$ and $\varepsilon_1\alpha = \alpha\varepsilon_2$.

Let $e_1, \ldots, e_n$ be the standard idempotents in H and put $H_i := e_iHe_i = K[\varepsilon_i]/(\varepsilon_i^{c_i})$, for $1 \leq i \leq n$. An H-module M is called locally free, if e_iM is a free H_i-module, for each i. Given a locally free module M, we define its rank vector $\mathbf{rk}(M) \in \mathbb{Z}^n$, by $\mathbf{rk}(M) := (\mathrm{rk}_{H_i}(e_iM))$. One has the following characterization of locally free modules.

Lemma. *The following conditions are equivalent for an H-module M.*

(1) *M is locally free.*
(2) $\mathrm{pdim}_H(M) \leq 1$.
(3) $\mathrm{idim}_H(M) \leq 1$.

2.2. **The second class of algebras.** We put

$$\hat{H} = \hat{H}(C, D, \Omega) := \varprojlim_k H(C, kD, \Omega).$$

Note that $\hat{H} = \hat{KQ}/\hat{I}$, where $\hat{KQ}$ is the completed path algebra of Q and $\hat{I}$ is generated by the relations

$$\varepsilon_i^{f_{j,i}}\alpha_{i,j}^{(g)} = \alpha_{i,j}^{(g)}\varepsilon_j^{f_{i,j}},$$

for $(i,j) \in \Omega$ and $1 \leq g \leq g_{i,j}$. If i is a vertex, then we put $\hat{H}_i := e_i\hat{H}e_i = K[[\varepsilon_i]]$. An $\hat{H}$-module M is called locally free, if e_iM is a free $\hat{H}_i$-module, for each i.

Put $\varepsilon := \sum_{i=1}^n \varepsilon_i^{c_i}$. Note that ε is a central element of $\hat{H}$ and $\hat{H}/\varepsilon \simeq H$. We have the following characterization of locally free $\hat{H}$-modules.

Lemma. *An $\hat{H}$-module M is locally free if and only if M is a free $K[[\varepsilon]]$-module.*

The algebra $\hat{H}$ has the following properties.

Lemma. *We have the following.*

(1) *$\hat{H}$ is Noetherian.*

(2) $\operatorname{gldim} \hat{H} \leq 2$.
(3) $\hat{H}$ *is a free* $K[[\varepsilon]]$*-module of finite rank.*

2.3. **The third class of algebras.** Let $K((\varepsilon))$ be the field of fractions of $K[[\varepsilon]]$ and put

$$\tilde{H} = \tilde{H}(C, D, \Omega) := K((\varepsilon)) \otimes_{K[[\varepsilon]]} \hat{H},$$

i.e. $\tilde{H}$ is the localization $\hat{H}_\varepsilon$. Note that $\tilde{H}$ is a finite dimensional $K((\varepsilon))$-algebra.

Let c be the least common multiple of $c_1, \ldots, c_n$. If we identify $K((\varepsilon))$ with $K((\delta^c))$, then it is possible to describe $\tilde{H}$ as a $K((\varepsilon))$-species, using the fields $K((\delta^{\frac{c}{c_i}}))$, for $1 \leq i \leq n$, and the bimodules $K((\delta^{\frac{c}{c_i}}, \delta^{\frac{c}{c_j}}))$, for $(i, j) \in \Omega$.

Observe that $H = T_S(B)$, $\hat{H} = T_{\hat{S}}(\hat{B})$ and $\tilde{H} = T_{\tilde{S}}(\tilde{B})$ for appropriate bimodules B, $\hat{B}$ and $\tilde{B}$, and

$$S := \prod_{i=1}^n K[\varepsilon_i]/(\varepsilon_i^{c_i}), \qquad \hat{S} := \prod_{i=1}^n K[[\varepsilon_i]] \quad \text{and} \quad \tilde{S} := \prod_{i=1}^n K((\varepsilon_i)).$$

3. Two functors

Let $\operatorname{mod}_{\text{l.f.}}(H)$ and $\operatorname{mod}_{\text{l.f.}}(\hat{H})$ be the categories of locally free H- and $\hat{H}$-modules, respectively. We have the functors $\operatorname{Loc}\colon \operatorname{mod}_{\text{l.f.}}(\hat{H}) \to \operatorname{mod}(\tilde{H})$ and $\operatorname{Red}\colon \operatorname{mod}_{\text{l.f.}}(\hat{H}) \to \operatorname{mod}_{\text{l.f.}}(H)$, given by

$$\operatorname{Loc} := \tilde{H} \otimes_{\hat{H}} - \qquad \text{and} \qquad \operatorname{Red} := H \otimes_{\hat{H}} -.$$

Then Red is dense (but neither full nor faithful), Loc is dense and faithful (but not full). Both Red and Loc do not respect isoclasses and Red does not respect indecomposable modules. If M is a locally free $\hat{H}$-module, then $\operatorname{Red}(M)$ is a locally free H-module and

$$\mathbf{rk}(M) = \mathbf{rk}(\operatorname{Red}(M)) = \mathbf{dim}(\operatorname{Loc}(M)).$$

Observe that $\hat{H}$ is an $K[[\varepsilon]]$-order in $\tilde{H}$ and the locally free $\hat{H}$-modules are exactly the $\hat{H}$-lattices.

4. Results

Theorem (Geiß/Leclerc/Schröer, 2019)**.** *The functors* Red *and* Loc *induce bijections between*

- *the indecomposable locally free rigid* H*-modules,*
- *the indecomposable locally free rigid* $\hat{H}$*-modules,*
- *the indecomposable rigid* $\tilde{H}$*-modules.*

They also induce isomorphisms between the exchange graphs of τ*-tilting modules.*

If M is an indecomposable rigid locally free H-module, then we put

$$\overline{M} := M / \sum_{f \in \operatorname{rad} \operatorname{End}_H(M)} \operatorname{Im}(f).$$

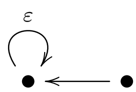
Theorem (Geiß/Leclerc/Schröer, 2019)**.** *We have a bijection between the indecomposable locally free rigid H-modules and the real Schur roots for (C, Ω) given by*

$$M \mapsto \mathbf{rk}(M).$$

Similarly, we have a bijection between the indecomposable locally free rigid H-modules and the real Schur roots for (C^T, Ω) given by

$$M \mapsto \dim(\overline{M}).$$

We illustrate the above theorem by the following example. Let H be the path algebra of the quiver



bound by $\varepsilon^2 = 0$. Then we have 4 indecomposable locally free rigid H-modules M_1, M_2, M_3 and M_4, which can be visualized as

$$\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}, \qquad 2, \qquad \begin{smallmatrix} 2 \\ 1 \\ 1 \end{smallmatrix} \qquad \text{and} \qquad \begin{smallmatrix} 2 & & \\ 1 & & 2. \\ & 1 & \end{smallmatrix}$$

The corresponding rank vectors are

$$(1,0), \qquad (0,1), \qquad (1,1) \qquad \text{and} \qquad (1,2).$$

The modules $\overline{M}_1$, $\overline{M}_2$, $\overline{M}_3$ and $\overline{M}_4$ can be visualized as

$$1, \qquad 2, \qquad \begin{smallmatrix} 2 \\ 1 \\ 1 \end{smallmatrix} \qquad \text{and} \qquad \begin{smallmatrix} 2 \\ 1 \end{smallmatrix},$$

and their dimension vectors are

$$(1,0), \qquad (0,1), \qquad (2,1) \qquad \text{and} \qquad (1,1).$$