THE COHOMOLOGY OF THE FOMIN–KIRILLOV ALGEBRA ON 3 GENERATORS

BASED ON THE TALK BY ESTANISLAO HERSCOVICH

1. General goal

There is a family $\mathrm{FK}(n)$, $n \geq 2$, of algebras defined over a field $\mathbb{K}$, which are quadratic, finite dimensional for $n = 2, 3, 4, 5$, and (braided) Hopf algebras. On the other hand, the algebras $\mathrm{FK}(n)$ are not Koszul due to a result of Ross from 1999.

Etingof and Ostrik have formulated the following conjecture.

Conjecture (Etingof, Ostrik, 2004)**.** *If A is a finite dimensional Hopf algebra, then $H^\bullet(A, \mathbb{K}) := \mathrm{Ext}^\bullet_A(\mathbb{K}, \mathbb{K})$ is a finitely generated algebra.*

Ştefan and Vay have proved the following.

Theorem (Ştefan, Vay, 2016)**.** *The algebra $H^\bullet(\mathrm{FK}(3), \mathbb{K})$ is finitely generated.*

The aim of this talk is to present a simpler proof of this result.

2. Quadratic algebras

An algebra A is called quadratic if $A = TV/\langle R\rangle$, for a finite dimensional vector space V over $\mathbb{K}$, where TV is the tensor algebra and $R \subseteq V^{\otimes 2}$. In the above situation we put

$$A^! := T(V^*)/\langle R^\perp\rangle,$$

where

$$R^\perp := \{\alpha \in (V^*)^{\otimes 2} \simeq (V^{\otimes 2})^* \mid \alpha(R) = 0\}.$$

Theorem. *There is a morphism of graded algebras*

$$A^! \to \mathrm{Ext}^\bullet_A(\mathbb{K}, \mathbb{K})$$

with the image $\bigoplus_{i\in\mathbb{N}} \mathrm{Ext}^{i,-i}_A(\mathbb{K}, \mathbb{K})$, where i is the cohomological degree and $-i$ is the internal degree.

For a quadratic algebra A we define the Koszul complex $K_\bullet(A)$ with $K_n(A) := (A^!_{-n}) \otimes A$ and $d_n\colon K_n(A) \to K_{n-1}(A)$ given by the multiplication by ι, where ι is the inverse image of Id_V under the canonical isomorphism $V^* \otimes V \to \mathrm{End}(V)$ (note that $V^* \otimes V \subseteq A^! \times A$).

Theorem. *$K_\bullet(A)$ is a subcomplex of the minimal projective resolution of $\mathbb{K}$.*

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3. Fomin–Kirillov algebras

For $n \geq 2$, let $V(n)$ be the vector space spanned by the elements $[i, j]$, with $1 \leq i < j \leq n$, modulo the subspace spanned by the elements $[i, j] + [j, i]$. We have an action and a coaction of the group $\mathbb{S}_n$ on $V(n)$ given by the formulas

$$\sigma \cdot [i, j] := [\sigma(i), \sigma(j)] \qquad [i, j] \mapsto (i, j) \otimes [i, j],$$

which give $V(n)$ a structure of a Yetter–Drinfeld left module over $\mathbb{K}\mathbb{S}_n$.

Let $R(n)$ be the subspace of $V(n)^{\otimes 2}$ spanned by the elements:

- $[i, j][i, j]$, for $1 \leq i < j \leq n$,
- $[i, j][j, k] + [j, k][k, i] + [k, i][i, j]$, if $i \neq j \neq k \neq i$,
- $[i, j][k, l] - [k, l][i, j]$, if $\#\{i, j, k, l\} = 4$.

We put $\mathrm{FK}(n) := T(V(n))/\langle R(n)\rangle$.

Theorem (Milinski, Schneider, 2000)**.** $\mathrm{FK}(n)$ *is a Hopf algebra in the category of Yetter–Drinfeld modules.*

Using the above theorem one shows.

Theorem (Mastnak, Pevtsova, Schauenburg, Witherspoon, 2010)**.** *The algebra* $\mathrm{Ext}^{\bullet}_{\mathrm{FK}(n)}(\mathbb{K}, \mathbb{K})$ *is braided graded commutative.*

4. Cohomology of FK(n)

Theorem (Ştefan, Vay, 2016)**.** *Let* $A = \mathrm{FK}(3)$*. Then* $\mathrm{Ext}^{\bullet}_A(\mathbb{K}, \mathbb{K}) \simeq A^{!}[\omega]$*, where* ω *has cohomological degree* 4 *and internal degree* -6.

For the proof we need two propositions. The first one is the following.

Proposition. *We have*

$$H_n(K_{\bullet}(A)) = \begin{cases} \mathbb{K}(-2n) & \text{if } n = 0, 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $(-)$ *denotes the shift in internal degree.*

The above proposition is proved by brute force calculations. The next one is the following.

Proposition. *The minimal projective resolution of* $\mathbb{K}$ *is given by the complex* $P_{\bullet}$ *such that*

$$P_n := \bigoplus_{0 \leq i \leq \lfloor \frac{n}{4} \rfloor} \omega_i K_{n-4i}(A),$$

where ω_i *is of internal degree* $6i$*, with* $\delta_n \colon P_n \to P_{n-1}$ *given by*

$$\delta_n(\omega_i \rho) := \omega_i d_{n-4i}(\rho) + \omega_{i-1} f_{n-4i}(\rho),$$

for some $f_m \colon K_m(A) \to K_{m+3}(A)$*,* $m \in \mathbb{N}$*, of internal degree* 6.

An idea of the proof. The minimal projective resolution of $\mathbb{K}$ starts with the sequence

$$0 \to \mathrm{Ker}(d_3) \to K_3(A) \to K_2(A) \to K_1(A) \to K_0(A) \to \mathbb{K} \to 0.$$

Moreover, by the first proposition we have a sequence

$$0 \to \mathrm{Im}(d_4) \to \mathrm{Ker}(d_3) \to \mathbb{K} \to 0,$$

while the sequence

$$\cdots \to K_7(A) \to K_6(A) \to K_5(A) \to K_4(A) \to \mathrm{Im}(d_4) \to 0$$

is acyclic. Using the horseshoe lemma, we get

$$P_4 \simeq K_4(A) \oplus K_0(A), \qquad P_5 \simeq K_5(A) \oplus K_1(A),$$
$$P_6 \simeq K_6(A) \oplus K_2(A) \qquad \text{and} \qquad P_7 \simeq K_7(A) \oplus K_3(A).$$

We continue inductively in this way and obtain the claim. □

Corollary. *As a graded vector space*

$$\mathrm{Ext}^\bullet_A(\mathbb{K},\mathbb{K}) := \mathrm{Hom}_A(P_\bullet,\mathbb{K}) \simeq A^![\omega].$$

In order to describe the algebra structure we observe we have an exact sequence

$$0 \to K_\bullet(A) \to P_\bullet \xrightarrow{\Omega} P_\bullet[4](-6) \to 0.$$

Then Ω is both $\mathbb{S}_3$-invariant and coinvariant, and by a result of Mastnak, Pevtsova, Schauenburg and Witherspoon this implies that Ω belongs to the center of $\mathrm{Ext}^\bullet_A(\mathbb{K},\mathbb{K})$. This consequently means that the map $\mathrm{Ext}^\bullet_A(\mathbb{K},\mathbb{K}) \to \mathrm{Ext}^\bullet_A(\mathbb{K},\mathbb{K})$ induced by Ω is injective, which in turn gives that the isomorphism in the corollary is the isomorphism of algebras.